

$L^1$ and $L^\infty$ Uniform Convergence of a Difference Scheme for a Semilinear Singular Perturbation Problem

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Summary. A nonlinear difference scheme is given for solving a semilinear singularly perturbed two-point boundary value problem. Without any restriction on turning points, the solution of the scheme is shown to be first order accurate in the discrete $L^1$ norm, uniformly in the perturbation parameter. When turning points are excluded, the scheme is first order accurate in the discrete $L^\infty$ norm, uniformly in the perturbation parameter.

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1. Introduction

We consider the problem

\[ \varepsilon y''(x) + a(x) y'(x) - d(x, y) = 0, \quad 0 < x < 1 \]

\[ y(0) = A, \quad y(1) = B \]

where $A, B$ are given constants, $\varepsilon$ is a parameter in $(0, 1]$, $a(\cdot) \in C^1([0, 1])$, $d \in C^1([0, 1] \times \mathbb{R})$ and

\[ d_y(x, y) \geq \delta > 0 \quad \text{on} \quad [0, 1] \times \mathbb{R} \]

\[ d_y(x, y) + a'(x) \geq \delta > 0 \quad \text{on} \quad [0, 1] \times \mathbb{R}, \]

where $\delta$ is independent of $x$ and $y$.

Note that no restriction is placed on the zeroes of $a(\cdot)$. This problem is easily seen to have a unique solution ($\S$ 2).

Using a Petrov-Galerkin finite element method on a uniform mesh of width $h$ we generate a nonlinear difference scheme for the problem. For $h$ sufficiently small (depending only on $a(\cdot)$ and $\delta$) we prove that this scheme has a unique

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solution. This solution is shown to be within $O(h)$ of the solution to (1.1) (1.2) (1.3) in the discrete $L^1$ norm, uniformly in $\varepsilon$ (see the end of this section for definitions of notation). Furthermore, if we also assume that the problem has no turning points, i.e., that $a'(\cdot)$ does not vanish anywhere in $[0, 1]$, we can in a natural way strengthen this result to $O(h)$ accuracy in the discrete $L^\infty$ norm. When (1.1) is linear with $C^2$ coefficients and $d(x, y) = b(x)y + f(x)$, an intermediate case is that of an isolated simple turning point: suppose $a(\frac{1}{2}) = 0$,

$$a(x) \neq 0 \quad \text{for} \quad x \neq \frac{1}{2}, \quad a'(\frac{1}{2}) > 0, \quad a'(x) \geq a'(\frac{1}{2})/2 \quad \text{for} \quad 0 < x < 1,$$

and $\lambda \equiv b(\frac{1}{2})/a'(\frac{1}{2})$ is not an integer. Then for $h = 1/N$ with $N$ even, the scheme is accurate of order $h^{\lambda_1}$ in the discrete $L^\infty$ norm, where $\lambda_1 = \min \{\lambda, 1\}$. In § 6 we present numerical evidence to substantiate these results.

Niijima [9] has recently considered the problem.

$$\varepsilon y''(x) - (a(x) y(x))' - b(x, y) = 0, \quad 0 < x < 1 \tag{1.4}$$

$$y(0) = \lambda, \quad y(1) = B$$

where $a(\cdot)$ and $b(\cdot, \cdot)$ are $C^2$ functions. He has derived a difference scheme for (1.4) whose solution (on a uniform mesh of width $h$) is within $O(h)$ of the solution to (1.4) in the discrete $L^1$ norm. (Our proof of the corresponding result for (1.1) (1.2) (1.3) imitates part of the argument of [9]).

Berger [1] has shown that the same scheme is $O(h)$ accurate in the discrete $L^\infty$ norm when the last condition of (1.4) is replaced by $a(\cdot) \geq \delta > 0$ on $[0, 1]$ and $b = 0$.

Lorenz [7] has considered (1.1) with $C^2$ coefficients, $a(\cdot) \geq \delta > 0$ and $d(\cdot, \cdot) \geq 0$. He has shown that a natural generalization of II'in's scheme [6] than yields $O(h)$ accuracy in the discrete $L^\infty$ norm.

In comparison, the analysis presented here requires less differentiability of the differential equation coefficients than do previous authors, and convergence results for the turning and nonturning point situations are obtained in a unifying framework.

The structure of the paper is as follows. Section 2 contains existence, uniqueness and a priori bounds for the solution of (1.1). Our difference scheme is generated in § 3 using finite elements. The $L^1$ result is proved in § 4 via the contraction mapping principle. We then obtain in § 5 a formula for the nodal errors in terms of discretized Green's functions. Substituting the $L^1$ bound into this formula and using the hypothesis $a(x) \neq 0, x \in [0, 1]$ we easily derive our $L^\infty$ error bound.

**Notation.** Throughout this paper $C$ denotes a generic positive constant independent of $x, y, h$ and $\varepsilon$. We say a quantity $g$ is $O(h)$ if $|g| \leq Ch$ for all $x, h$ and $\varepsilon$. If $z = (z_0, \ldots, z_N) \in \mathbb{R}^{N+1}$, define its discrete $L^1$ norm to be $\|z\|_1 = h \sum_{i=0}^{N} |z_i|$ and its discrete $L^\infty$ norm to be $\|z\|_\infty = \max_{0 \leq i \leq N} |z_i|$. We use $\tilde{y}$ to denote the $(N+1)$-dimensional vector $(y_0, y_1, \ldots, y_N) \equiv (y(x_0), y(x_1), \ldots, y(x_N))$ where $x_i = ih$. 