Fast and Precise Exciton Model Calculations of Nuclear Reactions

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Conversion of a set of differential master equations of a preequilibrium decay exciton model into a system of algebraic equations for mean lifetimes is discussed. The connection of the master equations with the closed-form expression is shown. A simple and computationally efficient method for obtaining the mean lifetimes of exciton states is presented.

1. Introduction

The exciton model of nuclear reactions (see Ref. 1 for a review) is rather successful in explaining particle spectra, excitation functions and other quantities at excitation energies of nuclei of the order of tens MeV. The nuclear reaction is described by a system of coupled differential Pauli master equations in this model. Such a system has been solved directly by numerical methods [2]. However, direct numerical integration of master equations is rather tedious. Sometimes, simple estimates of the mean lifetimes of individual exciton states are made, which lead to the closed formulations of the exciton model. The accuracy of the closed-form expressions is not always easy to control.

Physically measurable quantities are determined from the mean lifetimes of exciton states, i.e. from time-integrated occupation probabilities. To obtain them we need not to solve the system of differential master equations. They are given by a linear algebraic system of time-integrated master equations. This fact was first pointed out in Reference 3, and later expressed by several authors more explicitly [4-6].

In the present paper we want to discuss some aspects of the set of time-integrated master equations. In Section 2 we recapitulate some necessary formalism. The connection of the closed formulations of the exciton model with the master equations is shown in Section 3. Section 4 suggests a simple procedure for obtaining the mean lifetimes of exciton states. The procedure is computationally very efficient and gives the possibility of performing complicated exciton model calculations.

2. Master Equations

The process of equilibration in the nuclear reaction is described in the exciton model by a system of coupled master equations [7, 8]

\[ \frac{dP_n(t)}{dt} = P_{n+2}(t) \lambda_{n+2}^+ + P_{n-2}(t) \lambda_{n-2}^- - P_n(t) \frac{1}{\tau_n}, \]

where

\[ \tau_n = (\lambda_n^+ + \lambda_n^- + L_n)^{-1}. \]

The occupation probability of the \( n \)-exciton state at time \( t \) is denoted by \( P_n(t) \), the quantities \( \lambda_n^+ \) and \( \lambda_n^- \) are the transition rates for the process leading from \( n \) to \( (n+2) \) and \( (n-2) \) exciton states, respectively, and \( L_n \) is the total emission rate of the \( n \)-exciton state. The energy parameter is not written explicitly. All the rates are time-independent.

The system (1) is solved with the initial conditions

\[ P_n(0) = D_n. \]

It follows from the presence of the emission term \( L_n \) in (1) that the occupation probabilities \( P_n(t) \to 0 \) for the time \( t \to \infty \).
Knowing the probabilities $P_n(t)$ we can describe the time evolution of the reaction. However, only the time-integrated quantities are measurable at present. They are given by the mean lifetimes $T_n$—the times that the system spends in the $n$-exciton state

$$T_n = \int_0^\infty P_n(t) \, dt.$$ 

Integrating the system (1) over the time variable we immediately get the system of linear algebraic equations for the mean lifetimes

$$-D_n = T_{n-2} \lambda_{n-2}^+ + T_{n+2} \lambda_{n+2}^- - T_n \frac{1}{\tau_n},$$

(2)

To obtain the lifetimes $T_n$ and thus all measurable quantities the system of differential equations (1) need not be solved but it is sufficient to solve the much simpler algebraic system (2).

Summing equations (2) over all $n$'s one gets

$$\sum_n D_n = \sum_n T_n L_n.$$  

(3)

Here, the left hand side expresses the number of nuclei created at a start of a reaction. The right hand side describes the number of decayed nuclei. Equation (3) demonstrates thus the conservation of nuclei in the reaction.

The single-value starting configuration $D_n = \delta_{nn_0}$ is usually applied to (2). It is, however, desirable to discuss the general form of $D_n$'s in Equations (2). If we investigate the problem of multiparticle and gamma emission in the exciton model the original set of master equations can be divided into subsets which can be solved successively. The subsets differ by the excitation energy and/or relate to different composite nuclei. Each subset has the form (1) with the additional source term describing the flux of the nuclei into the given exciton state by decay at preceding reaction steps. In the time-integrated form we get for each subset system of the form (2) with the total flux $D_n$ obtained from the solution of equations for preceding reaction steps.

3. Closed Formulations of the Exciton Model

In the discussion of this section we assume the simple starting configuration $D_n = \delta_{nn_0}$.

Several simple formulas exist for the estimates of the mean lifetimes $T_n$ in the exciton model. All these should be deduced from the system of algebraic equations (2). The simplest estimate $T_n = 1/\lambda_n^+$ follows from (2) when neglecting the quantities $\lambda_n^-$ and $L_n$.

The best of the closed formulas seems to be that of Wu and Chang [9]

$$T_n \approx \tau_n \left( \prod_{i=n_0}^{n-2} \lambda_i^+ \tau_i \right) \left( 1 + \lambda_n^+ \tau_n \lambda_{n+2}^- \tau_{n+2}^- \right).$$

(4)

One obtains this estimate taking the solution of an approximate system

$$-\delta_{n_0} = T_{n-2}^{[0]} \lambda_{n-2}^+ - T_{n+2}^{[0]} \frac{1}{\tau_n}$$

as a zeroth approximation, i.e.

$$T_n^{[0]} = \tau_n \left( \prod_{i=n_0}^{n-2} \lambda_i^+ \tau_i \right)$$

(5)

and then once simply iterating the system according to [10]

$$T_n^{[i+1]} = \tau_n \left( T_n^{[i]} \lambda_n^+ \lambda_{n-2}^- + T_{n+2}^{[i]} \lambda_{n+2}^- + D_n \right).$$

(6)

In the computer code STAPRE [11] another method is used for the calculation of the lifetimes $T_n$, which describes the sequence of steps occurring during the process of the equilibration. According to that

$$T_n = \tau_n \sum_k b_n^{[k]}$$

and $b_n^{[k]}$ are calculated from recurrence relations

$$b_n^{[k+1]} = b_{n-2}^{[k]} \tau_{n-2}^- \lambda_{n-2}^- + b_{n+2}^{[k]} \tau_{n+2}^- \lambda_{n+2}^-.$$  

This method is mathematically obviously just the simple iteration method (6) for the system (2) rewritten in a difference scheme*. As a zeroth approximation, the estimate

$$b_n^{[0]} = \delta_{nn_0}$$

is taken, which is less convenient than the estimate (5). Therefore, more iterations are needed.

4. Solution of Time-Integrated Master Equations

Luider [5] solves the system of algebraic equations (2) directly using the standard computer procedure. He reports the computer time is saved by a factor of 6 when compared to the numerical integration of the differential system (1). Another method of solution of system (2) is given in Reference 4 which takes advantage of the special tridiagonal form of the system.

* The transition rate $\lambda_n^+$ is introduced for the transitions with $\Delta N = 0$ in Reference 11. In fact, this term has no influence in the method of this Reference.