Energy Levels of QED in a Euclidean Formulation

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Hydrogen-like energy levels of scalar and spinor QED are calculated using a Euclidean functional approach. The matter field is integrated over. Stationary points of the resulting effective action already yield a hydrogen like level structure for the energy. There is an interesting difference between the scalar and the spinor case. Whereas for spinors the conventional results are reproduced, the calculation for scalars yields a fine structure which is opposite in sign to the conventional one and has no critical singularity at $Z \rho = 8/3$. The crucial structural difference between the two cases is that, for scalars, minima for the gauge invariant energy are not extrema of the action, even for time independent fields.

1. Introduction

Recently, the crucial role of judiciously chosen configurations in classical Euclidean field theories as important contributions to physical quantities expressed as functional integrals has been restressed. In this note we should like to discuss a special feature of Euclidean gauge theories which arises from the fact that in the transition to the Euclidean regime, along with the replacement $t \rightarrow it$, a substitution $A_0 \rightarrow -iA_0$ for the gauge fields has to be performed. There is an interesting difference between gauge theories of scalar and spinor matter fields. In particular, we find that the relativistic fine structure of the "hydrogen like" levels for scalar QED (e.g. pionic atoms) seems to be opposite in sign to the one obtained from the conventional Klein-Gordon equation in an external Coulomb field. Related to this we find that there is no singularity at $Z \rho = 8/3$, where the conventional external field calculation breaks down. On the other hand, for spinor fields where the gauge field enters linearly in the equations of motion, the Euclidean approach yields the same result as the conventional calculation.

The paper is organized in four sections. In the following section we give a qualitative account of the problem. Then, the "hydrogen like" levels of scalar and spinor QED are extracted from the Euclidean partition functional integral. Our results are discussed in a concluding section.

2. Euclidean and Minkowskian Formulation

The essential point which we want to convey is already clear if we confine ourselves to time-independent classical configurations with

$$\label{2.1} A_\mu(t, \vec{x}) = (\phi(\vec{x}), \vec{0}).$$

The Lagrangian for scalar QED then is

$$\label{2.2} L = -\frac{1}{2\kappa} \int d^3x \left\{ -(V\phi)^2 + (V\phi)^2 + m^2 \phi^2 - \phi^2 \phi^2 \right\},$$

where $\phi(\vec{x})$ is a scalar field of mass $m$. The corresponding Euclidean Lagrangian

$$\label{2.3} L_E = -\frac{1}{2\kappa} \int d^3x \left\{ (V\phi)^2 + (V\phi)^2 + m^2 \phi^2 + \phi^2 \phi^2 \right\}$$

is a negative definite functional on a suitable space of real functions and coincides with the energy component of the gauge invariant Belinfante energy momentum tensor. Hence, extrema of the Euclidean action are also extrema of the energy.

However, the Euler Lagrange equations for $L$ and $L_E$ differ, in spite of the time-independence:

$$L: \quad \Delta \phi - \phi^2 \phi = 0$$

$$(-\Delta + m^2 - \phi^2) \phi = 0,$$
\[ L_E: -\Delta \phi_E + \phi_E^2 \phi_E = 0 \]
\[ (-\Delta + m^2 + \phi_E^2) \phi = 0. \]  
\[(2.5)\]

Although, for solutions \((\phi, \phi)\) and \((\phi_E, \phi_E)\) of (2.4) and (2.5) we have

\[ L(\phi, \phi) = L_E(\phi, \phi) \]
and

\[ L(\phi_E, \phi_E) = L_E(\phi_E, \phi_E) \]
\[(2.6)\]
solutions of (2.5) are not extrema of the action and, therefore, in general do not solve the equation of motion. In general, we have

\[ L_E(\phi_E, \phi_E) \geq L_E(\phi, \phi). \]  
\[(2.7)\]

Of course, to verify the statements the existence of non-trivial solutions with finite action* has to be established [1].

The difference of Equations (2.4) and (2.5) lies in the difference of the sign with which the potential enters into the matter field equations. This will be responsible for the change in the fine-structure mentioned in the introduction.

The case of spinor QED requires a more refined discussion which we postpone to the end of the next section.

3. Hydrogen-Like Levels in QED

In our approach, we integrate over the matterfield \(x\) in the functional integral

\[ Z = \int \cdots \mathcal{D}x \, e^{i\mathcal{S}[\cdots x]}, \]

thus incorporating the matter field quantum fluctuations from the beginning. One is left with an effective action for the electromagnetic field. Quantum fluctuations of the latter around a stationary point of this action appear in a loop expansion. At least in a weak coupling limit they will be shown to give only small corrections to the energy levels already obtained in the "classical" approximation for this action.

a) Scalar QED

Our starting point is the Euclidean functional integral

\[ W = \frac{1}{\mathcal{N}} \int \mathcal{D} A \mathcal{D} \psi \mathcal{D} \bar{\psi} \ e^{-\frac{i}{\hbar} \int \mathcal{L} \ dx^4 \left( \partial_A A + \bar{\psi} \gamma^0 \gamma^5 \psi + m^2 \psi \right)} \]
\[(3.1)\]

where \( A_\mu = \partial_\mu - ie A_\mu \) and \( D^2 = -D_0^2 - \bar{D}^2 \) and the gauge \( \partial_\mu A_\mu = 0 \) was chosen. For large \( T = t_2 - t_1 \), \( W \) behaves as

\[ W \sim e^{E_g \tau} \]  
\[(3.2)\]

and \( E_g \) is the ground state energy, other energy levels can be obtained [2] as poles in the Laplace transform of \( W \) with respect to \( T \).

After integration over the scalar matter field \( \psi \) we get

\[ W = \frac{1}{\mathcal{N}} \int \mathcal{D} A_\mu e^{S_{\text{eff}}[A_\mu]} \]

\[ S_{\text{eff}}[A_\mu] = -\frac{1}{2} \int_{t_1}^{t_2} dt \int dx (\partial_\mu A_\mu)^2 - \int d^4 x \text{tr} \log \frac{D^2 + m^2}{D_0^2}, \]
\[(3.3)\]

\( \Delta_0 \) is a subtraction term to be specified below.

The idea is to expand the effective action around a time independent reference field:

\[ A_\mu = A_\mu^{(0)} + \eta_\mu \]
\[ A_\mu^{(0)} = (\phi(\tilde{x}), \tilde{\theta}); \]
\[ S_{\text{eff}} = S_{\text{eff}}[A_\mu^{(0)}] \]

\[ + \int d^4 x \frac{\delta S_{\text{eff}}}{\delta A_\mu(x)} [A_\mu^{(0)}] \eta_\mu(x) \]

\[ + \frac{1}{2} \int d^4 x \int d^4 x' \eta_\mu(x) \frac{\delta^2 S_{\text{eff}}}{\delta A_\mu(x) \delta A_\mu(x')} [A_\mu^{(0)}] \eta_\mu(x') \]
\[ + \cdots \]
\[(3.5)\]

and to require \( S_{\text{eff}} \) to be stationary at this point. This means

\[ \frac{d S_{\text{eff}}}{\delta A_\mu(x)} [A_\mu^{(0)}] = \delta_{\mu 0} \Delta \phi(\tilde{x}) - \langle x | \frac{1}{D^2 + m^2} j_\mu | x \rangle | A_\mu^{(0)} = 0 \]
\[(3.6)\]

with

\[ j_\mu = -ie(\bar{D}_\mu - \bar{D}_\mu) \]

or more explicitly, introducing

\[ |x\rangle = |\tilde{x}\rangle e^{\mu x} \]

and eigenfunctions

\[ (\bar{D}^2 + m^2) \varphi_n(\tilde{x}) = \lambda_n \varphi_n(\tilde{x}) \]
\[ \bar{D}^2 (\phi) = (\omega - e \phi)^2 - \Delta \]
\[(3.7)\]

we get for \( \mu = 0 \)

\[ \Delta = \sum_n \frac{1}{2 \pi} \int d\omega \frac{1}{\lambda_n(\omega)} e^{(\omega - e \phi)} \frac{1}{\| \varphi_n \|^2} \]
\[(3.8)\]