COMPONENT PARTIALLY ORDERED SETS

V. V. Pashenkov

Properties of component partially ordered sets (i.e., dense subsets of Boolean algebras) are used to construct mappings of Boolean algebras generalizing the idea of homomorphisms; the properties of a minimal Boolean algebra generated by a given component partially ordered set are investigated.

Structural operations in a partially ordered (p.o) set $P$ will be denoted by the symbols $\supseteq$, $\subseteq$, $\cup$, $\cap$. The symbols "\cup" and "\cap" denote set-theoretic operations.

Definition 1. The principal ideal of a p.o. set $P$ generated by an element $p \in P$ will be denoted by $p^+$. A subset $I \subseteq P$ is called a semi-ideal if, for any $p \in P$, the inclusion $p \in I$ implies $p^+ \subseteq I$. The set of all semi-ideals of a p.o. set $P$, including the empty semi-ideal $\emptyset$ and ordered by inclusion, will be denoted by $L(P)$.

It is easily seen that the p.o. set $L(P)$ is a complete infinitely distributive lattice, and so ([1], p. 210) it possesses relative pseudocomplements. We denote the pseudocomplement of an element $I \in L(P)$ by $I^*$. An element $I \in L(P)$ is called closed, if $I^{**} = (I^*)^* = I$. The subset consisting of all closed elements of the lattice $L(P)$ is denoted by $K(P)$. The Glivenko-Stone theorem implies that $K(P)$ is a complete Boolean algebra, and the mapping

$$\varphi: L(P) \rightarrow K(P),$$

where $\varphi(I) = I^{**}$, is a homomorphism of $L(P)$ on $K(P)$. We isomorphically imbed $P$ in $L(P)$ by $\varepsilon: P \rightarrow L(P)$, where $\varepsilon(p) = p^+$ for any element $p \in P$. The image of the set $\varepsilon(P)$ under the mapping $\varphi$ is denoted by $\varphi(\varepsilon P)$, and the smallest subalgebra of the algebra $K(P)$ containing the set $\varphi(\varepsilon P)$ is denoted by $S(P)$.

Definition 2. Subsets $X$ and $Y$ of the p.o. set $P$ are called disjunctive (written $X \sqcup Y$) if, for any $x \in X$ and $y \in Y$, we have $x^+ \cap y^+ = \emptyset$. The largest subset disjunctive to a subset $X$ is called the disjunctive complement of $X$ and is denoted by $X^d$. A subset $X \subseteq P$ is called a component subset if $X^{dd} = (X^d)^d = X$.

Remark 1. It is easily seen that, for any semi-ideal $I \in L(P)$, the pseudocomplement $I^*$ coincides with its disjunctive complement $I^d$ in $P$. Hence the Boolean algebra $K(P)$ is the set of components of the p.o. set $P$ ordered by inclusion.*

Definition 3. A set $B$ of a Boolean algebra $A$ is called complete, if it does not contain the zero element $O_A$ of $A$ and, for any element $a \in A$, $a \neq O_A$, there is an element $b \in B$ such that $b \supseteq a$.

Let $B$ be a subset of the Boolean algebra $A$. We write $(B)^A$ for the set of all elements of $A$ which are exact upper bounds in $A$ for all possible subsets $\{b_\alpha\} \subseteq B$ and the element $O_A$.

$$(B)^A = \{O_A\} \cup \{a \in A: a = \sup \{b_\alpha\} \text{ for some set } \{b_\alpha\} \subseteq B\}.$$  

Similarly we make the definition

*See [2], p. 5.

(B)_{\text{fin}}^A = \{O_A\} \cup \{a \in A: a = \sup_{i=1}^{n} \{b_i\} \text{ for some finite subset } \{b_i\} \subseteq B\}.

**Lemma 1.** A subset $B$ is complete in a Boolean algebra $A$ if and only if $(B)_{\text{fin}}^A = A$ and $B \subseteq A \setminus \{O_A\}$.

**Proof.** The sufficiency of the condition is obvious. To prove necessity, we show that, if $B$ is dense in $A$, then for any $a \in A$, $a \neq O_A$, we have

$$a = \sup_{a \in B} a.$$ 

If we assume the contrary, there is an upper bound $c$ in the set $a^+ \cap B$ such that $c \neq a$. Then $ac' \neq O_A$, and so there is an element $b \in B$ such that $ac' \geq b$. This, however, is impossible since $b \leq a$, i.e., $b \in a^+ \cap B$: hence $b = c$ and $b = c'$ which implies that $b = O_A \notin B$.

**Lemma 2.** For any p.o. set $P$ we have

$$[\varphi (vP)]^K(P) = K(P),$$ 

and so it follows from Lemma 1 that the set $\varphi (vP)$ is complete in the Boolean algebra $K(P)$.

For the proof it suffices to note that any nonempty closed semi-ideal $I \subseteq K(P)$, $I = \{p_{\alpha}\}$, can be expressed in the form

$$I = \sup \{\varphi (v (p_{\alpha}))\}.$$ 

**Theorem 1.** If $P$ is a p.o. set, the following conditions are equivalent:

1) For any elements $q \neq p$ of a p.o. set $P$, there is an element $r \leq q$, such that $r^+ \cap p^+ = \Phi$;

2) Any principal ideal of a p.o. set $P$ is a component of $p$;

3) The mapping $\varphi : L(P) \rightarrow K(P)$ isomorphically imbeds the subset $\varepsilon(P)$ in $L(P)$ in the Boolean algebra $K(P)$;

4) There is a Boolean algebra $A$ containing a complete subset isomorphic to the p.o. set $P$.

**Proof.** 1 $\Rightarrow$ 2. Let $p$ and $q$ be any elements of the p.o. set $P$, $q \neq p ^+$. Then there is an element $r \in P$ and $q \leq r$ and $r^+ \cap p^+ = \Phi$; hence $r \in p_{\varphi}$. It follows that $r \notin p_{\varphi}d$, and so $q \notin p_{\varphi}d$. By the same token we can prove that $p_{\varphi}d \subseteq p^+$ for any $p \in P$. Since the inverse inclusion is obvious, we have $p_{\varphi}d = p^+$.

2 $\Rightarrow$ 3. This follows directly from remark 1.

3 $\Rightarrow$ 4. The set $\varphi (vP) \subseteq K(P)$ is isomorphic to the p.o. set $P$, and Lemma 2 implies that it is dense in $K(P)$.

4 $\Rightarrow$ 1. Suppose that $P$ is contained in the Boolean algebra $A$. Let $p$, $q \in P$ and $q \neq p$. Then the element $q^p$ of the Boolean algebra is not the zero element. Hence there is an element $r \in P$ such that $r \leq q^p \leq p$. It is easily seen that $r$ is the element required in condition 1.

**Definition 4.** A p.o. set satisfying one of the conditions of Theorem 1 is called a component set.

**Theorem 2.** Let $P$ be a component partially ordered set contained as a complete subset in the Boolean algebra $A$. Then there exist isomorphisms $\alpha$ and $\beta$, complete over $P$:

$$S(P) \xrightarrow{\beta} A \xrightarrow{\alpha} K(P).$$

**Proof.** It is known [4] that complete Boolean algebras are determined up to isomorphisms by their complete subsets. Hence the completion $A$ of an algebra $A$, in which it is easily verified that $P$ is also complete, is isomorphic over $P$ to the Boolean algebra $K(P)$:

*Use of the symbols $\sup$ and $\inf$, implies the existence of the corresponding bounds.
†Here and in the sequel we write $c'$ for the complement of $c$ in the algebra $A$.
‡Condition 1 is the same as condition (a) in [3], p. 63. The equivalence of conditions 1 and 4 is proved in [3].
**In considering component partially ordered sets $P$, we assume the $P$ itself is in $S(P)$ and $K(P)$ and identify $p$ with $\varepsilon (P)$.