NONUNIFORM ESTIMATE OF THE REMAINDER TERM IN THE INTEGRAL LIMIT THEOREM

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We shall deal with the sequence of independent identically distributed random variables

\[ \xi_1, \xi_2, \ldots, \xi_n, \ldots \] 

with distribution function \( F(x) \) and characteristic function \( f(t) \).

Let \( F_n(x) \) be the distribution function of the normalized sum

\[ S_n = n^{-\alpha} \sum_{i=1}^{n} \xi_i \] 

while \( f_n(t) \) is the corresponding characteristic function.

We assume that in the case \( \alpha > 1 \) the expected value of random variables \( \xi_i \) is zero.

We denote by \( G_\alpha(x) \) the distribution function of the stable law with exponent \( \alpha \) (\( 0 < \alpha \leq 2, \alpha \neq 1 \)) and characteristic function

\[ \varphi_\alpha(t) = e^{-t^\alpha} \left( 1 - 2\alpha \right) \frac{\cos \alpha \frac{\pi}{2} t}{\sin \alpha \frac{\pi}{2} t} \].

We denote the pseudomoment and the absolute pseudomoment by \( \mu(m) \) and \( \nu(m) \) respectively, where

\[ \mu(m) = \int x^m d \left( F(x) - G_\alpha(x) \right) \],

\[ \nu(m) = \int |x|^m d \left( F(x) - G_\alpha(x) \right) \] .

Let \( r = 1 + \lfloor \alpha \rfloor \), where \( \lfloor \alpha \rfloor \) is the integral part of \( \alpha \).

THEOREM 1. If there exists \( \nu(1 + \alpha) \), and if \( \mu(0) = \ldots = \mu(r) = 0 \), then, when \( n > 2 \), we have the estimates:

\[ |F_n(x) - G_\alpha(x)| \leq \frac{C'}{n^\alpha (1 + |x|)} \max \left\{ \nu(1 + \alpha) \nu(1 + \alpha)^{\frac{1}{2\alpha}} \right\} \]

when \( 0 < \alpha < 1 \), and

\[ |F_n(x) - G_\alpha(x)| \leq \frac{C''}{n^\alpha (1 + |x|)^{\frac{\alpha}{2}}} \max \left\{ \nu(1 + \alpha) \nu(1 + \alpha)^{\frac{1}{2\alpha}} \right\} \]

for \( 1 < \alpha < 2 \), where \( C' \) and \( C'' \) are constants depending on \( \alpha \).

For the proof of the theorem just formulated, we need the following lemmas. For their formulation we use the notation:

\[ T_1 = n^\alpha \nu(1 + \alpha)^{-1}, \quad T_2 = n^\alpha \nu(1 + \alpha)^{-\frac{1}{2\alpha}}, \quad T_3 = n^\alpha \nu(1 + \alpha)^{-\frac{1}{3\alpha}}. \]

LEMMA 1 (L. Osipov). Let $M(x)$ be a non-decreasing function, $N(x)$ a function of bounded variation, and let $\tilde{\Delta}(x) = M(x) - N(x)$. Let $A$ and $T$ be positive constants, and let $s \geq 1$ be an integer. If

1) $\tilde{\Delta}(-\infty) = \tilde{\Delta}(\infty) = 0$,

\[ \int |x|^r \, d\tilde{\Delta}(x) < \infty; \]

2) function $N(x)$ has a derivative and

\[ |N'(x)| \leq \frac{A}{(1+|x|)^r}, \]

then

\[ (1+|x|)^r |\tilde{\Delta}(x)| \leq C_r(t) \left[ \int_0^T \left| \frac{\delta(t)}{t} \right| \, dt + \int_0^T \left| \frac{\delta_3(t)}{t} \right| \, dt + \frac{A}{T} \right], \tag{8} \]

where $\delta(t)$ and $\delta_3(t)$ are the respective Fourier–Stieltjes transforms of functions $\tilde{\Delta}(x)$ and $x^3 \tilde{\Delta}(x)$.

LEMMA 2 (L. Osipov). Let $W(x)$ be a function of bounded variation and $\hat{w}(t)$ its Fourier–Stieltjes transform. If

1) $W(-\infty) = W(\infty) = 0$,

\[ \int |x|^r \, dW(x) < \infty \]

for some integral $s \geq 1$, then

\[ \int e^{itx} \, d \left( x^s W(x) \right) = s!(-it)^{-s} \sum_{k=0}^s \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \hat{w}(t). \tag{9} \]

LEMMA 3. If the conditions of Theorem 1 are met, we then have the estimate

\[ |f_n(t) - f_n(t)| \leq \frac{e^{-\frac{1}{n^4} |t|^s}}{n^s} \left\{ \begin{array}{ll} \frac{1}{4} |t|^s & \text{for } |t| \leq T_1, \; v(1+\alpha) \geq 1, \\ \frac{1}{4} |t|^s |v(1+\alpha)|^{-\frac{s}{2}} & \text{for } |t| \leq T_2, \; v(1+\alpha) < 1. \end{array} \right. \tag{10} \]

LEMMA 4. If the conditions of Theorem 1 hold then, for $n > 1$, we have the following bound

\[ \left| \frac{d}{dt} \left( f_n(t) - f_n(t) \right) \right| \leq \frac{\sqrt{e^{-\frac{1}{n^4} |t|^s}}}{n^s} \left\{ \begin{array}{ll} \frac{1}{4} |t|^s & \text{for } |t| \leq T_1, \; v(1+\alpha) \geq 1, \\ \frac{1}{4} |t|^s |v(1+\alpha)|^{-\frac{s}{2}} & \text{for } |t| \leq T_2, \; v(1+\alpha) < 1. \end{array} \right. \tag{11} \]

LEMMA 5. If the conditions of Theorem 1 are met then, for $\alpha > 1$ and $n > 2$, we have the bound

\[ \left| \frac{d^2}{dt^2} \left( f_n(t) - f_n(t) \right) \right| \leq \frac{e^{-\frac{1}{n^4} |t|^s}}{n^s} \left\{ \begin{array}{ll} \frac{1}{4} |t|^s & \text{for } |t| \leq T_1, \; v(1+\alpha) \geq 1, \\ \frac{1}{4} |t|^s |v(1+\alpha)|^{-\frac{s}{2}} & \text{for } |t| \leq T_2, \; v(1+\alpha) < 1. \end{array} \right. \tag{12} \]

The proofs of Lemma 1 and 2 are given in [1]. We note that Lemma 1 in [1] is proven for $s \geq 2$, but its validity for $s \geq 1$ also readily follows.

The proof of Lemma 3 is given in note [3] (Lemmas 2 and 4).

Proof of Lemma 4. Let $0 < t \leq T_1$ and

\[ \Delta = \left| \frac{d}{dt} \left( f_n(t) - f_n(t) \right) \right| = n \left| \frac{1}{n^s} f_n \left( \frac{1}{n^s} t \right) - \phi_n^{-\frac{1}{n}} \left( n^s \frac{1}{n^s} t \right) \right| < n \left| \frac{1}{n^s} f_n \left( \frac{1}{n^s} t \right) - \phi_n^{-\frac{1}{n}} \left( n^s \frac{1}{n^s} t \right) \right| + 
\]

\[ + \left| \phi_n^{-\frac{1}{n}} \left( n^s \frac{1}{n^s} t \right) f_n \left( \frac{1}{n^s} t \right) \phi_n \left( n^s \frac{1}{n^s} t \right) \right| = n (\Delta_1 + \Delta_2), \tag{13} \]

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