A Path Following Algorithm for a Class of Convex Programming Problems

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Abstract: We present a primal-dual path following interior algorithm for a class of linearly constrained convex programming problems with non-negative decision variables. We introduce the definition of a Scaled Lipschitz Condition and show that if the objective function satisfies the Scaled Lipschitz Condition then, at each iteration, our algorithm reduces the duality gap by at least a factor of 

\[(1 - \delta/\sqrt{n}),\]

where \(\delta\) is positive and depends on the curvature of the objective function, by means of solving a system of linear equations which requires no more than \(O(n^3)\) arithmetic operations. The class of functions having the Scaled Lipschitz Condition includes linear, convex quadratic and entropy functions.

Key words: Convex Programming, Primal-dual Methods, Path Following, Interior Point Algorithm, Entropy Optimization.

1 Introduction

The nonlinear optimization problem considered in this paper is the linearly constrained convex programming problem which has the following form:

\[(LCCP) \quad \min f(x) \quad \text{s.t.} \quad Ax = b, \quad x \geq 0 \quad (1.1)\]

where \(f(x)\) is a real convex function satisfying a new condition, termed a Scaled Lipschitz Condition (whose definition will be given in §3). \(A\) is an \(m \times n\) matrix with rank \(m\), and \(x \in \mathbb{R}^n\), \(b \in \mathbb{R}^m\).

For the case that \(f(x)\) is linear or convex quadratic, many solution techniques are known. One such method, known as the primal-dual path following algorithm, has been developed by Monteiro and Adler ([14], [15]). The basic idea can be traced back to late 60's, see Fiacco and McCormick [9]. With this method, each iterate is generated close to the so-called “central path”. This allows for a

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(1 - $\Omega(1/\sqrt{n})$) factor reduction in the duality gap at each iteration and only requires solving a system of linear equations at each iteration.

Using path following methods to solve a general (LCCP) problem has been studied by many contributors, among them, Kortanek, Potra and Ye [12]. In their primal-dual path following algorithm the duality gap is also reduced by a factor of $(1 - \Omega(1/\sqrt{n}))$ at each iteration, but each iteration requires the solution of a system of nonlinear equations. Generally speaking, solving a system of nonlinear equations may be harder than solving (LCCP) itself.

In this paper, we extend some results [12] for a special class of LCCP problems where the objective function $f(x)$ satisfies an additional condition, called the Scaled Lipschitz Condition, which we show is less restrictive than certain other smoothness conditions appearing in the literature. With this condition we then present a path following algorithm which both reduces the duality gap by a factor of $(1 - \delta/\sqrt{n})$, where $\delta$ is positive and depends on the curvature of the objective function, and also only requires solving a linear system of equations at each iteration.

Jarre [11] and Hertog, Roos and Terlaky [10] developed similar results for solving another class of convex programming problems where both the objective function and the inequality constraints are convex functions whose Hessian matrices satisfy the so-called Relative Lipschitz Condition. The authors indicate that this condition may be difficult, in general, to check for a given problem.

This paper is organized as follows. In §2, some of Kortanek, Potra and Ye's results is presented in a slightly different way, and a new proof for one of their results is given. In §3, we present an algorithm for a class of LCCP problems in which $f(x)$ satisfies the Scaled Lipschitz Condition. In §4, we discuss some possible applications, especially to the linearly constrained Entropy optimization problems. In §5, we compare the Scaled Lipschitz Condition with some of the other conditions in the literature. In §6, we present our conclusions.

\section{Path Following Method for General (LCCP) Problems}

In this section, we state the Lagrangian dual problem corresponding to LCCP and define the associated primal-dual central path. We present our main algorithm next.

The Lagrangian dual for continuously differentiable functions is the following.

\[(D) \max f(x) - Vf(x)^T x + b^T y \]
\[\text{s.t. } s = Vf(x) - A^T y \geq 0 \]

where $s \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $Vf(x)$ is the gradient of $f(x)$. 