Hadronic Corrections to QCD Sum Rules and Light Quark Masses

C.A. Dominguez
Departamento de Fisica, Universidad Técnica Federico Santa Maria, Valparaiso, Chile, and
Institut für Physik der Johannes-Gutenburg Universität, D-6500 Mainz, Federal Republic of Germany

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Abstract. A parametrization of the $J^P = 0^-$ hadronic continuum, in the framework of Extended PCAC, is discussed with emphasis on finite-width effects and on the constraints imposed by the correct threshold behavior of the pion spectral function. As an application light quark masses are calculated using both Hilbert and Laplace transform QCD sum rules. The results for the running quark masses are: $(\bar{m}_u + \bar{m}_d)_{1\text{GeV}} = 16 \pm 2 \text{ MeV}$, $(\bar{m}_u + \bar{m}_u)_{1\text{GeV}} = 199 \pm 27 \text{ MeV}$, and a ratio $R = 2(\bar{m}_u + \bar{m}_d)/(\bar{m}_u + \bar{m}_d) = 25 \pm 4$.

1. Introduction

Considerable effort has been devoted in recent years to the calculation of chiral-symmetry breaking parameters [1, 2] in the framework of QCD sum rules [3–11]. In particular, the use of these sum rules has allowed for direct estimates of light quark masses thus representing a major step forward from current algebra which by itself only yields quark mass ratios [2, 12]. With a few exceptions, though, the emphasis has been on improving the QCD information e.g. by calculating higher order radiative corrections, by including non-perturbative contributions of increasing dimensionality or by searching for particular sum rules which would enhance this aspect of the problem and suppress, to some conjectured degree, the hadronic continuum. There is no doubt that the progress made in understanding the QCD aspect of the problem is very impressive but unfortunately it has not been accompanied by an equally strong effort in the hadronic sector. The assumptions made in the literature concerning hadronic continuum contributions cover a wide range extending from the reasonable to the purely ad-hoc and the present situation calls for a more detailed and accurate quantitative analysis.

A suitable dynamical framework for estimating these hadronic contributions is that of Extended PCAC (EPCAC) first proposed in [13], developed later in [14–15] and recently rediscovered by a number of authors. The EPCAC hypothesis amounts to a saturation of dispersion relations involving the pion (kaon) spectral function by $3\pi(\pi\pi K)$ resonances or radial excitations either in the zero-width [13] or finite-width [15] approximation. A distinctive advantage of this approach is that it has already been tested in a wide variety of applications and shown to accurately predict chiral-symmetry breaking corrections to PCAC in a systematic fashion. By using EPCAC as input in QCD sum rule calculations one could then be reasonably certain that the results do not represent more than a test of the input assumptions.

In this paper we discuss in some detail an EPCAC parametrization of the hadronic continuum in a manner suited for QCD sum rule applications with particular emphasis on the finite-width approximation. In particular, we find that the constraint imposed by the correct threshold behavior of the pion spectral function is quite restrictive and it affects the values of quark masses by a non-negligible amount. We then proceed to calculate light quark masses using both Hilbert transform and Laplace transform QCD sum rules. As these two sum rules place different emphasis on the hadronic continuum, comparison of the results from both methods will provide a reasonable indication of the true uncertainties in the EPCAC parametrization.

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2 Permanent Address
2. EPCAC Parametrization of the Hadronic Continuum

We begin by defining the two-point function

$$\psi_s(q^2) = i \int d^4x e^{iqx} \langle 0 | T(\partial_\mu A_\mu(x) \partial^\nu A_{\nu}^+(0)) | 0 \rangle,$$

where the axial-vector currents carry pion quantum numbers (most of our subsequent discussion translates literally into the kaon case). Using PCAC, i.e.

$$\partial_\mu A_\mu(x) = \sqrt{2} f_\pi \mu_\pi^2 \phi_\pi(x),$$

where $$f_\pi = 93.2 \text{ MeV},$$ one finds for the imaginary part

$$\frac{1}{\pi} \text{Im} \psi_s(t) = 2 f_\pi^2 \mu_\pi^4 \left[ \delta(t - \mu_\pi^2) + \rho_\pi(t) \right],$$

where the pion spectral function $$\rho_\pi(t)$$ defined through the pion propagator ($$q^2 = -Q^2$$)

$$A_\pi(Q^2) = \frac{1}{\mu_\pi^2 + Q^2 + \int_0^\infty dt \frac{\rho_\pi(t)}{t + Q^2}}.$$  

The threshold behavior of $$\rho_\pi(t)$$ is known from current algebra (in the chiral limit) [16] and thus

$$\frac{1}{\pi} \text{Im} \psi_s(t) \mid_{t=0} = 2 f_\pi^2 \mu_\pi^4 \left[ \delta(t - \mu_\pi^2) \right].$$

It has been checked that taking the threshold at $$t = 0$$ introduces only negligible errors as none of the integrals that concern us exhibit infrared singularities. The constraint imposed by (5) clearly rules out zero-width resonance models, at least qualitatively, and we shall discuss later its quantitative impact.

In the spirit of EPCAC we parametrize the pion spectral function by an infinite series of three-pion resonances. Considering first the zero-width approximation one readily finds

$$\frac{1}{\pi} \text{Im} \psi_s(t) = 2 f_\pi^2 \mu_\pi^4 \sum_{n=1}^\infty \frac{1}{\rho_\pi(t)} \left[ \delta(t - \mu_\pi^2) + \rho_\pi(t) \right],$$

where

$$\rho_\pi(t) = \rho_\pi(t),$$

and

$$f_\pi^2 = \mu_\pi^2 + 2M_\rho^2,$$

For the mass spectrum and the (chiral-symmetry breaking) heavy pion decay constants we have chosen the framework of the dual model [13, 15]. This model predicted long ago [13] the existence of a $$\pi'$$ with $$\mu_\pi \approx 1.1 \text{ GeV},$$ in reasonable agreement with recent experimental evidence [17]. Also, the pion propagator calculated in this model accurately reproduces (at $$Q^2 = 0$$) dispersion theoretic results [16] and its validity in the space-like region has been amply confirmed by several analyses of one-pion-exchange hadronic cross section data [18]. In this way the free parameter $$\beta$$ in (8) was consistently determined to be $$\beta \approx 2.5.$$ Concerning the time-like region relevant to QCD sum rule applications we would expect the model to remain reliable below the second radial excitation ($$Q^2 < 2 - 2.5 \text{ GeV}^2$$) i.e. the region where the $$3\pi$$ channel is dominant. As one approaches higher hadronic thresholds a smooth continuum is expected to gradually build up to yield the correct linearly rising asymptotic behavior of $$\text{Im} \psi_s(t).$$

This can be incorporated into the model in the standard fashion by adding e.g. the perturbative continuum term [4, 19]

$$\frac{1}{\pi} \text{Im} \psi_s(t) \mid_{\text{pert}} = \frac{3}{8 \pi^2} \left[ \bar{m}_u + \bar{m}_d \right]^2 \left(1 + \frac{17}{3 \pi} \right) t \theta(t - t_\phi).$$

where $$\bar{m}_i$$ are the running quark masses in the $$\overline{\text{MS}}$$ scheme and $$\bar{m}_i$$ is obtained from the extended resonance ansatz [20] for three flavors. Notice that the infinite series in (6) is very rapidly converging so that the main contribution comes from $$\pi',$$ a small amount from $$\pi'$$ and a negligible contribution from the rest of the resonances. However, it is important to realize that all radial excitations make a non-negligible global contribution as reflected in the value of $$\beta$$ in (8), i.e. $$\beta \approx 2.5.$$ In fact, in a model with only one resonance ($$\beta = 2$$) one would have $$|f_\pi| \approx 1.5 \text{ MeV},$$ while in reality $$|f_\pi| \approx 2.3 \text{ MeV}$$ for $$\beta \approx 2.5.$$ Furthermore, integrating (6) one finds that the single resonance model gives an area about 2.4 times smaller than the full model. Although the parameter $$\beta$$ is well under control from a variety of independent fits [18] this strong dependence is clearly an undesirable feature. However, the blame is not on the model but rather on the zero-width approximation. In fact, it will be shown next that when the correct threshold behavior is enforced this dependence on $$\beta$$ virtually disappears.

The zero-width parametrization can be unitarized in the standard fashion by shifting the poles from the real axis, i.e.

$$\pi^2 (\mu_\pi^2 - t) \rightarrow \frac{\gamma \mu_\pi^2}{(\mu_\pi^2 - t)^2 + \gamma^2 \mu_\pi^4},$$

where

$$\gamma = \frac{\Gamma_{\pi^2}}{2\mu_\pi^2} \approx 0.1 - 0.2.$$  

is obtained from the extended resonance ansatz [20] $$\Gamma_{\pi^2} \approx \gamma \mu_\pi^2$$ and the experimental data [17] on $$\pi'.$$ Notice, however, that the resulting Breit–Wigner expression (10) does not vanish at threshold. This typical feature of resonance models [21] is corrected by multiplying the spectral function by an appropriate factor in order to ensure the correct threshold behavior. In this way one finds

$$\frac{1}{\pi} \text{Im} \psi_s(t) \mid_{\text{unit}} = \frac{3}{8 \pi^2} \left[ \bar{m}_u + \bar{m}_d \right]^2 \left(1 + \frac{17}{3 \pi} \right) t \theta(t - t_\phi) \left(1 + \frac{17}{3 \pi} \right) t \theta(t - t_\phi) \left(1 - \frac{17}{3 \pi} \right) t \theta(t - t_\phi).$$

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