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De Rham Cohomology of an Analytic Space

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Introduction

It is well known that the sheaf Ω_X^* of germs of holomorphic differential forms on a complex analytic manifold X is a resolution of $\mathbb{C}$ on X. As a consequence, the complex cohomology $H^*(X,\mathbb{C})$ of X is isomorphic, when X is Stein, to the cohomology of the De Rham complex $\Gamma(X,\Omega_X^*)$ of global sections.

Recent examples show that if X is an analytic space, then Ω_X^*, as defined by Grauert and Grothendieck, need not be a resolution ([7] and [16]), so that no such isomorphism exists.

We show in this paper that, nevertheless, the complex cohomology of the analytic space X can still be obtained from the De Rham complex: there exists a canonical splitting

$$\mathbf{H}^*(X;\Omega_X^*)\simeq H^*(X,\mathbb{C})\oplus A^*$$

of the hypercohomology $\mathbf{H}^*(X;\Omega_X^*)$ of Ω_X^* into the classical cohomology and a second factor.

The result also holds in the real analytic and semianalytic cases, and for the complex of smooth ($\mathscr{C}^\infty$) differential forms. It is proved constructing a right inverse, by means of integration, to the canonical edge homomorphism $H^*(X,\mathbb{C})\to\mathbf{H}^*(X,\Omega_X^*)$ (Theorem 3.11).

As a corollary, one deduces the following complement to Grothendieck's results on the algebraic De Rham cohomology of regular algebraic schemes [9]. Let X be a complete variety (not necessarily regular) or a prescheme locally of finite type over $\mathbb{C}$ with only isolated singularities. Then the hypercohomology of the complex $\Omega_{a,X}^*$ of rational regular differential forms on X splits canonically (cf. 3.14):

$$\mathbf{H}^*(X,\Omega_{a,X}^*)\simeq H^*(X,\mathbb{C})\oplus A^*$$

Particular cases of these results have been announced in [12]. To our knowledge, Norguet [15] posed the question about the relationship between (smooth) De Rham and classical cohomologies of an analytic variety.

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1. Semianalytic Sets

We refer to [13] for proofs of the properties stated in this section.

1.1. Let $(X, \mathcal{O}_X)$ be a real analytic space, not necessarily reduced, and let $\mathcal{O}_{X_r}$ be the sheaf of analytic functions of X, that is, functions associated to sections of $\mathcal{O}_X$ on the open subsets of X. For each $x \in X$, let $S(x)$ be the smallest family of germs at x of subsets of X such that:

(1) $a, b \in S(x)$ implies $a \cup b \in S(x)$ and $a - b \in S(x)$;

(2) if f is an analytic function on a neighborhood of x, then the germ at x of the set $(f > 0)$ belongs to $S(x)$.

A subset M of X is called semianalytic if its germ at $x \in X$ belongs to $S(x)$, for all $x \in X$. The semianalytic sets of a complex analytic space are defined by considering the associated real analytic structure.

Let U be open in $\mathbb{R}^n$, $\mathcal{J}$ a coherent sheaf of ideals of the sheaf $\mathcal{O}_U$ of analytic functions on U and Y the set of zeros of $\mathcal{J}$. Then the semianalytic sets of the real analytic space $(Y, \mathcal{O}_U/\mathcal{J} | Y)$ are the semianalytic sets of $(U, \mathcal{O}_U)$ included in Y.

Locally finite unions and intersections, complements, closures, interiors and boundaries of semianalytic sets are semianalytic.

1.2. Suppose M is semianalytic in $(X, \mathcal{O}_X)$. A point $x \in M$ is q-simple (or q-regular), for q an integer $\geqq 0$, if there is a neighborhood U of x in M such that $(U, \mathcal{O}_{X_r} | U)$, is isomorphic (as a ringed space of local $\mathbb{R}$-algebras) to an open subspace $(V, \mathcal{O}_V)$ of $\mathbb{R}^q$; in particular the 0-simple points of M are the isolated points of M. The set of simple points of M (i.e., q-simple points for some q) is dense in M. The dimension, $\dim M$, of M is $\leqq p$ if there are no q-simple points of M with $q > p$; $\dim M = p$ if $\dim M \leqq p$ but not $\dim M \leqq p-1$.

Suppose $\dim M = p$. Then $\dim \bar{M} = p$ and $\dim(\bar{M} - M) < p$, where $\bar{M}$ is the closure of M in X. The semianalytic set $bM = \bar{M} - M$ is called the border of M; bM is closed if and only if M is locally closed. If M^* is the set of p-simple points of M, then $(M^*, \mathcal{O}_{X_r} | M^*)$ is a real analytic submanifold of X of dimension p, and $sM = M - M^*$ is semianalytic in X, $\dim sM < p$.

2. Semianalytic Chains

Throughout this section, K is a principal ideal domain and M is a closed semianalytic subset in the real analytic space $(X, \mathcal{O}_X)$. We allow the possibility that $M = X$.

If Φ is a family of supports in the locally compact space X and $\mathcal{F}$ is a sheaf of K-modules on X, then $H_*(X; \mathcal{F})(H_*^{\Phi}(X; \mathcal{F}))$ denotes Borel-Moore homology with coefficients in $\mathcal{F}$ and closed supports (supports in Φ) [2]. We will denote by K the constant sheaf with value K and we will sometimes use the notation $H_*(X)$ instead of $H_*(X; K)$. If $F \subset X$ is