UPPER BOUND FOR THE PRODUCT OF NONHOMOGENEOUS LINEAR FORMS

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It is proved that for any unimodular lattice $\Lambda$ with homogeneous minimum $L > 0$ and any set of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ there exists a point $(y_1, y_2, \ldots, y_n)$ of $\Lambda$ such that
\[
\prod_{1 \leq i \leq n} |y_i + \alpha_i| \leq e^{-n^2 \eta_n^2} (1 + n L^{(2n)}(\sqrt{n - 1} L^{(2n)}))^{-n/2},
\]
where $\gamma_n = n/(n-1)$.

1. INTRODUCTION

Here we consider a problem of the geometry of numbers, that of estimating from above the product of nonhomogeneous linear forms. Minkowski conjectured that, in the space $\mathbb{R}^n$, for any lattice $\Lambda$ and any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ there exists a point $Y = (y_1, y_2, \ldots, y_n) \in \Lambda$ such that
\[
\Pi(\Lambda) = \prod_{1 \leq i \leq n} |y_i + \alpha_i| \leq 2^{-n \det \Lambda}. \tag{1}
\]


At present, we know (see [4]) that for any $n$-dimensional lattice $\Lambda$ and any vector $\alpha \in \mathbb{R}^n$ there exists a vector $Y \in \Lambda$ such that
\[
\Pi(\Lambda) \leq 2^{-n^2} (3 + 10^{-4})^{-1} \eta_n^{-1} \det \Lambda, \tag{2}
\]
where $\lim_{n \to \infty} \eta_n = 2e - 1$.

Estimates for $\Pi(\Lambda)$ were given by Mordell [5, 6], but these yield to estimate (2) for sufficiently large dimension of the lattice $\Lambda$.

Papers [7, 8] deal with an upper bound for $\Pi(\Lambda)$ in terms of the homogeneous minimum of the lattice $\Lambda$. By the homogeneous minimum we mean the quantity $L = \min_{Y \in \Lambda, Y \neq 0} |y_1 y_2 \cdots y_n|$. In [7] these estimates are given only for $n = 3$, and in [8] for lattices of a special form.

In this present paper we give an upper bound for $\Pi(\Lambda)$ for large dimensions of a lattice $\Lambda$ whose homogeneous minimum satisfies the condition
\[
L \geq n^{-(8^3)\lambda}, \tag{3}
\]
where $0 < \lambda < 1$ and $\lambda$ is fixed relative to $n$. Under condition (3), our estimate is significantly better than (2).
For simplicity of exposition we will consider unimodular lattices, i.e., det $A = 1$; the homogeneous minimum will be denoted by $L$, and the nonhomogeneous minimum by $\Pi(A)$.

1. **THEOREM.** For any lattice $\Lambda$ (det $\Lambda = 1$) we have

$$\Pi(\Lambda) \leq 2^{-n/2} \gamma^{n} (1 + 3L^{8/(3n)}) (\gamma^{2} - 2L^{8/(2n)})^{-n/2},$$

where $\gamma = n^{1/(n-1)}$.

It is clear from (I) that

$$\gamma^{n} (1 + 3L^{8/(3n)}) (\gamma^{2} - 2L^{8/(2n)})^{-n/2} \rightarrow \infty, \quad n \rightarrow \infty,$$

under the condition (3). We first prove two lemmas.

**LEMMA 1.** The polyhedron $T$ with $2^n$ faces defined by the conditions

$$|m \sum_{i \leq m} x_i - m' \sum_{j \leq m'} y_j| \leq mm',
$$

$$|x_1 + x_2 + \ldots + x_n| \leq n,$$

where the sum extends over all decompositions $n = m + m'$ and where $i$ ranges over any set of $m'$ numbers and $j$ over any set of $m$ numbers from $(1, 2, \ldots, n)$, has volume $2^n$.

**Proof.** The body $T$ is a prism. If in the first inequality we put $x_n = -x_1 - x_2 - \ldots - x_{n-1}$, we obtain the parallelohedron studied by Voronoi (see [9]).

Let $\Phi(\Lambda_0)$ be the fundamental region of the principal lattice $\Lambda_0$. We will prove that the set $\Lambda_0 + (1/2)T$ completely covers all of $\mathbb{R}^n$, i.e., $\Phi(\Lambda_0) \subset (1/2)T$ and therefore $V((1/2)T) = 1$. Assume that $\Lambda_0 + (1/2)T$ is not a cover. Then there exists a point $z_0$ lying on a face of the body $(1/2)T$ and belonging only to $(1/2)T$, i.e.,

$$z_0 \notin (1/2)T + \gamma, \quad \gamma \neq 0, \quad \gamma \notin \Lambda_0.$$

We consider two cases.

**Case 1.** The point $z_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ lies on a "lateral" face of $(1/2)T$. We may assume (without loss of generality) that

$$0 \leq m \sum_{i \leq m} x_i^{(0)} - m' \sum_{j \leq m'} x_j^{(0)} = (1/2) mm'. \quad (4)$$

Consider the set $Y_{m'} + (1/2)T$, where $Y_{m'} = (-1, -1, \ldots, -1, 0, \ldots, 0)$ (each of the first $m'$ coordinates is equal to $-1$, the rest to $0$).

We will show that $z_0$ lies on a face of $Y_{m'} + (1/2)T$. Indeed, since any "lateral" face of $Y_{m'} + (1/2)T$ is defined by an equation

$$m \sum_i (x_i - 1) - m' \sum_j x_j = (1/2) mm',
$$

it follows from (4) that

$$|m \sum_{i \leq m} x_i^{(0)} - 1 - m' \sum_{j \leq m'} x_j^{(0)}| = |mm'/2 - mm'| = mm'/2.$$

Thus, $z_0 \notin Y_{m'} + (1/2)T$, which contradicts our assumption.

**Case 2.** The point $z_0$ lies on the base of the body $(1/2)T$. Then we have

$$x_1^{(0)} + x_2^{(0)} + \ldots + x_n^{(0)} = n/2. \quad (5)$$

Consider the set $Y_n + (1/2)T$, where $Y_n = (-1, -1, \ldots, -1)$. We will show that $z_0 \notin Y_n + (1/2)T$. Since the base of $Y_n + (1/2)T$ is defined by the equation

$$|x_1 - 1 + \ldots + (x_n - 1)| = n/2 - n = n/2,$$

it follows from (5) that

$$|x_1 - 1 + \ldots + (x_n - 1)| = n/2 - n = n/2;$$

thus, $z_0 \notin Y_n + (1/2)T$, which contradicts our assumption.