Using Lemma 2, it is now easy to show that the operator \( B \rightarrow f(U^{-1}, U) B \) is inverse to the operator

\[
e^{-\beta_0 A} e^{-\beta_0 B} - \lambda, \text{ so that } \lambda \not\in \sigma(e^{-\beta_0 A} e^{-\beta_0 B}).\]

Theorem 3 is proved.

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**LITERATURE CITED**

3. E. B. Dynkin, "Representation of the series \( \ln(e^X e^Y) \) in noncommuting \( x \) and \( y \) in terms of commutators," Mat. Sb., 25, No. 1, 155-162 (1949).

**APPROXIMATION OF THE FUNCTION \( \text{sign} \, x \) IN THE UNIFORM AND INTEGRAL METRICS BY MEANS OF RATIONAL FUNCTIONS**

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Estimates are obtained for the nonsymmetric deviations \( R_n[\text{sign} \, x] \) and \( R_n[\text{sign} \, x]_L \) of the function \( \text{sign} \, x \) from rational functions of degree \( \leq n \), respectively, in the metric

\[
C([-\delta, -\delta] \cup [\delta, \delta]), \quad 0 < \delta < \exp(-\alpha \sqrt{n}), \quad \alpha > 0,
\]

and in the metric \( L[-1, 1] \):

\[
R_n[\text{sign} \, x] \leq \exp(-\pi n/(2 \ln n)), \quad n \rightarrow \infty,
\]

\[
10^{-n} \exp(-2\pi \sqrt{n}) < R_n[\text{sign} \, x]_L < \exp(-\pi \sqrt{n^2/4} + 150).
\]

Let \( 0 < \delta < 1, \Delta(\delta) = [-1, -\delta] \cup [\delta, 1] \);

\[
R_n[f; \Delta(\delta)] = R_n[f] = \inf_{R(x) \Delta(\delta)} \max\{f(x) - R(x)\},
\]

\[
R_n[f; [-1, 1]]_L = R_n[f]_L = \inf_{R(x) [-1, 1]} \int_{-1}^{1} |f(x) - R(x)| \, dx,
\]

where \( R(x) \) is a rational function of order at most \( n \).

Bulanov [1] proved that for \( \delta \in [e^{-n}, e^{-1}] \) the inequality

\[
\exp\left(-\frac{\pi n}{2 \ln(1/\delta)}\right) \leq R_n[\text{sign} \, x] \leq 30 \exp\left(-\frac{\pi n}{2 \ln(1/\delta) + 4 \ln(1/\delta) + 4}\right)
\]

is valid. The lower estimate in this inequality was previously obtained by Gonchar ([2], cf. also [1]).

Here we sharpen estimate (1) on the right (cf. Sec. 1) and obtain corresponding estimates for \( R_n[\text{sign} \, x]_L \) (cf. Sec. 2). The sharpening of the right-hand side of estimate (1) is obtained by means of a more efficient (as compared with [1]) choice of interpolation points and poles of the rational function approximating \( \text{sign} \, x \).

1. UPPER ESTIMATE FOR $R_n[\text{sign } x]$

1.1. Preliminary Lemmas

**Lemma 1.** For any $x$ the double inequality

$$\max \{0, 1 - x^2/6\} \leq 2x/(e^x - e^{-x}) \leq 1$$

is valid.

**Proof.** It suffices to carry out the proof for $x \geq 0$ (in view of the evenness of all the functions). From the Maclaurin series expansion of the function $e^x$ we obtain

$$e^x - e^{-x} = 2x + 2x^3/3! + \ldots + 2x^{2k+1}/(2k+1)! + \ldots =$$

$$= 2x\left(1 + x^2/3! + \ldots + x^{2k}/(2k+1)! + \ldots \right) = 2x \varphi(x).$$

(2)

Thus,

$$2x/(e^x - e^{-x}) = 1/\varphi(x).$$

Moreover, $\varphi(x) \geq 0$ for $x \geq 0$, and if $x^2/6 < 1$, then bearing in mind that for $k = 1, 2, \ldots, 6k \leq (2k+1)!$, we obtain

$$\varphi(x) \leq 1 + x^2/6 + \ldots + (x^2/6)^t = (1 - x^2/6)^{-1};$$

(3)

if, on the other hand, $x^2/6 \geq 1$, then $1 - x^2/6 \leq 0$. In order to obtain the assertion of the lemma it suffices to apply inequality (3) to (2).

**Lemma 2.** For all $c > 0$, integers $n \geq 256c + 1/c$ and $k \in [e^{-2n}, e^{-\pi^2/e}]$ there exists a polynomial $P(x)$ of degree $n$ such that

$$\max_{x \in [a, e^{-1}\ln\delta]} \left| \frac{P(x)}{P(-x)} \right| \leq 6 \exp\left( \frac{2n^6}{2\ln \delta} + 4\pi^2c + \frac{5}{2c} \right).$$

**Proof.** Put $m = 4c[-\ln \delta]$. Since for $a \geq 1$, $[a] \geq a/2$, for $n \geq c^{-1}$ we have

$$2c \ln (1/\delta) \leq m \leq 4c \ln (1/\delta).$$

Put $N = n - 2m$, $M = [N/\ln(1/\delta) + 3/2]$, $\alpha_j = \delta^{n-j}/N$, $0 \leq j < N$, $P(x) = (x - \alpha_j)^{m} \prod_{j=0}^{N-1} (x - \alpha_j)$.

Clearly, the degree of the polynomial $P(x)$ does not exceed $2m + N = n$, and for $n \geq 2$ we have $1 \leq n/2 \leq N \leq n$.

We estimate $|P(x)/P(-x)|$ for $\delta = \alpha_0 \leq x \leq \alpha_{N-1}$. Let $\delta \leq \alpha_p < x < \alpha_{p+1}$, $0 \leq p < N - 2$. Then

$$\left| \frac{P(x)}{P(-x)} \right| = \left( \frac{x - \delta}{x + \delta} \right)^m \left( \frac{\alpha_{N-1} - x}{\alpha_{N-1} + x} \right)^m \prod_{j=0}^{N-1} \left( \frac{x - \alpha_j}{x + \alpha_j} \right)^m \prod_{j=p+1}^{N-1} \left( \frac{\alpha_j - x}{\alpha_j + x} \right)^m \prod_{j=0}^{N-1} \left( \frac{\alpha_{N-1} - x}{\alpha_{N-1} + x} \right)^m.$$

We have

$$\left( \frac{x - \delta}{x + \delta} \right)^m \leq \frac{1}{\delta^{(p+1)/m}} \delta((p+1)/(m+1)) = \exp\left( m \ln \delta + \delta^{-(p+1)/m} \right) \leq \exp\left( \frac{m^2}{\ln \delta} \right).$$

Since $(2t - 1)m / \ln \delta \leq 2N$ for $0 < \delta < \exp[-(n/c)^{1/2}]$ and the values of $N, m, t \geq 1$ defined by us, we have

$$\left( \frac{x - \delta}{x + \delta} \right)^m \leq \exp\left( \frac{m^2}{\ln \delta} \right).$$

(4)

Analogously,

$$\left( \frac{\alpha_{N-1} - x}{\alpha_{N-1} + x} \right)^m \leq \left( \frac{1 - \delta^{(p+1)/m}}{1 + \delta^{(p+1)/m}} \right)^m = \exp\left( m \ln \delta \right).$$

453