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Equivariant Homology and Mackey Functors

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Let G be a finite group. The G-equivariant homology and cohomology theories which one meets in nature (e.g. bordism, cobordism, K-theory, stable homotopy) are usually defined at least for all subgroups K of G and there are restriction and induction relations among the various theories. The appropriate framework to formalize the relations is the axiomatic representation theory of Green [6] and the theory of Mackey functors of Dress [5, 5a].

We consider equivariant homology theories which have a Mackey structure (see definition below). Such theories are in particular modules over the Burnside ring $\Omega(G)$ of finite G-sets. Our main theorem is a splitting theorem for theories localized at a prime ideal of $\mathbb{Z}$ and a geometric description of the theories localized at a prime ideal of $\Omega(G)$.

The main theorem contains various results of Conner, Floyd and Stong [11] on bordism. It generalizes the localization theorems for equivariant homology or cohomology (Segal [9], tom Dieck [2, 3]) and has a variety of applications to the theory of characteristic numbers for G-manifolds. Since the main theorem explains itself as a basic result about equivariant homology we describe numerous applications at another occasion.

Our theorem sheds light on the determination of the prime ideal spectrum of $\Omega(G)$ (Dress [4]), this being in a sense a special case of our splitting theorem. In another paper we describe its applications to the determination of prime ideals in cohomology theories.

Our presentation uses the language of Mackey functors introduced by Dress. I have tried to use only published results. So I had to recall some of his terminology to make the paper readable for topologists.

1. Mackey Structures on Equivariant Homology Theories

Let G be a finite group and let $G^\wedge$ be the category of finite G-sets and G-maps. A *Mackey functor* $\mathfrak{M}=(\mathfrak{M}_*,\mathfrak{M}^*)$ on $G^\wedge$ is a pair of functors $\mathfrak{M}_*: G^\wedge \to \mathrm{Ab}$ (covariant), $\mathfrak{M}^*: G^\wedge \to \mathrm{Ab}$ (contravariant) into the category Ab of abelian groups which agree on objects and which have the following properties (M 1) and (M 2). We write $\mathfrak{M}(X)=\mathfrak{M}_*(X)=\mathfrak{M}^*(X)$ for objects X in $G^\wedge$ and $\mathfrak{M}_*(\varphi)=\varphi_*$, $\mathfrak{M}^*(\varphi)=\varphi^*$ for maps φ in $G^\wedge$.

(M 1) If

$$\begin{tikzcd} X \arrow[r, "\bar{\varphi}"'] \arrow[d, "\bar{\psi}"'] & X_2 \arrow[d, "\psi"] \\ X_1 \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

is a pull-back diagram in $G^\wedge$ then the diagram

$$\begin{tikzcd} \mathfrak{M}(X) \arrow[r, "\bar{\varphi}_*"'] & \mathfrak{M}(X_2) \\ \mathfrak{M}(X_1) \arrow[r, "\varphi_*"'] \arrow[u, "\bar{\psi}^*"] & \mathfrak{M}(Y) \arrow[u, "\psi^*"'] \end{tikzcd}$$

is commutative.

(M 2) If $X \xrightarrow[i]{} Z \xleftarrow[j]{} Y$ is a sum diagram in $G^\wedge$ then

$$\mathfrak{M}(X) \xleftarrow[i^*]{} \mathfrak{M}(Z) \xrightarrow[j^*]{} \mathfrak{M}(Y)$$

is a product diagram in Ab.

The notion of Mackey functor is due to Dress [2]. A more explicit statement of the axioms is given in Green [6].

Let G-Top^2 be the category of pairs (X, A) of numerable G-spaces, with A closed in X, and G-maps. (A G-space X is called numerable if it has a numerable covering $(U_j \mid j \in J)$ by G-sets U_j such that each U_j admits a G-map into some homogeneous G-space G/U, U subgroup of G. See [3].)

A *G-homology theory* t_* shall be for the purpose of this paper a sequence $(t_n : G\text{-Top}^2 \to \mathrm{Ab} \mid n \in \mathbb{Z})$ of covariant functors and a sequence of natural transformations $(\partial_n \mid n \in \mathbb{Z})$

$$\partial = \partial_n = \partial_n(X, A) : t_n(X, A) \to t_{n-1}(A, \emptyset) = : t_{n-1}(A)$$

such that the usual axioms hold in the following form: (1) G-homotopic maps induce the same homomorphisms. (2) The long homology sequence is exact. (3) The excision isomorphism $t_n(X, A) \cong t_n(X - U, A - U)$ holds, provided there exists a continuous G-invariant function $\tau : X \to [0, 1]$ with the properties $U \subset \tau^{-1}(0)$, $X - A \subset \tau^{-1}(1)$. (4) The theory is additive, i.e. the functors t_n preserve sums. In particular: If $X_1 \subset X_2 \subset X_3 \subset \cdots$ is a sequence of G-cofibrations then the canonical map

$$\operatorname{colim} t_*(X_i) \to t_*(\operatorname{colim} X_i)$$

is an isomorphism.

If S is a finite G-set then we define

$$tS_n(X, A) : = t_n(S \times X, S \times A).$$