Piecewise Polynomial Taylor Methods for Initial Value Problems

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Abstract. A class of explicit Taylor-type methods for numerically solving first-order ordinary differential equations is presented. The basic idea is that of generating a piecewise polynomial approximating function, with a given order of differentiability, by repeated Taylor expansion. Sharp error bounds for the approximation and its derivatives are given along with a stability analysis.

1. Introduction

This paper deals with the numerical solution of a single first-order ordinary differential equation

\begin{align*}
    u'(x) &= f(x, u(x)), \quad a \leq x, \\
    u(a) &= u_0
\end{align*}

on a finite interval \([a, b]\), although the ideas carry over to single higher-order equations as well as to systems of first-order equations. Assuming \(f(x, u)\) to be sufficiently differentiable, we construct by repeated Taylor expansion a piecewise polynomial function \(y(x)\) which is of degree \(m \geq 1\) and which belongs to the class \(C^p[a, b]\) where \(0 \leq p \leq m - 1\). The function \(y(x)\) provides a continuous approximation to the exact solution \(u(x)\) as well as to certain of its derivatives.

The method presented here is a generalization of a well known numerical method which Davis [1, Chpt. 9] calls continuous analytic continuation and Henrici [2, Chpt. 2] calls the Taylor expansion method. To distinguish our method from the one just mentioned, and to emphasize its polynomial nature, we shall call our technique a piecewise polynomial Taylor method. The Taylor expansion method, which produces a piecewise polynomial approximation of class \(C^0[a, b]\), coincides with our method when \(p = 0\). The present method, on the other hand, allows for construction of smoother approximations \(y \in C^p[a, b], p \geq 1\), for example \(m\)-th degree splines where \(p = m - 1\). Unfortunately, increased smoothness is gained at the expense of both accuracy and stability.

After presenting our method in §2, we show in §3 that it is of order \(O(h^{m-p})\) in the step size \(h\). Curiously enough, the \(j\)-th derivatives also have error bounds

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of order $O(h^{m-p})$ for $0 \leq j \leq p$. Thus, the rates of convergence of $y(x)$ and its lower derivatives are reduced for smoother approximations. The higher derivatives of $y(x)$ have error bounds of order $O(h^{m-j+1})$ for $p+1 \leq j \leq m$. In §4 it is shown that the method is absolutely stable for sufficiently small $h$ only for the least smooth approximations, i.e., $p=0$. Smoother approximations are only weakly stable. The results of §3 and §4 are made explicit by a numerical example in §5. Then comparisons of the piecewise polynomial Taylor method with the piecewise polynomial methods of Loscalzo and Talbot [3–5] are given in §6.

The results of this paper are largely negative in that the new methods, i.e., those corresponding to $p \geq 1$, turn out to be only weakly stable. However, even in the case of the well-known Taylor expansion method, the error bounds of Theorem 1 appear to be new. For example, when $p=0$ the error in the $j$-th derivative at the mesh points is of the order $O(h^m)$ for $0 \leq j \leq m$.

2. The Piecewise Polynomial Taylor Method

Suppose that $f(x, v)$ is $t$ times continuously differentiable with respect to both $x$ and $v$ in some domain $D$ of the $(x, v)$-plane where $t \geq 1$. Assume that $D$ contains the graph of the exact solution $u(x)$ of (1)–(2) on $[a, b]$, so that $u \in C^{t+1}[a, b]$. Then let $m$ and $p$ be integers such that $1 \leq m \leq t$ and $0 \leq p \leq m-1$, and let the mesh points in $[a, b]$ be defined by $x_i = a + ih$, $0 \leq i \leq N$, where $N = (b-a)/h$ and $h$ is the step size.

Further assume that $f(x, v)$ is known and that the following functions, defined recursively from $f(x, v)$, are also known:

\begin{align}
(3) \quad f^{(0)}(x, v) &= f(x, v), \\
(4) \quad f^{(j)}(x, v) &= \frac{\partial}{\partial x} f^{(j-1)}(x, v) + f(x, v) \frac{\partial}{\partial v} f^{(j-1)}(x, v), \quad 1 \leq j \leq m - 1.
\end{align}

For $v = u(x)$, it follows from the differential equation (2) that the $f^{(j)}$ are total derivatives with respect to $x$ of $f$, and consequently they are derivatives of $u$ of one higher order, i.e.

\begin{align*}
f^{(j)}(x, u(x)) &= \frac{\partial}{\partial x} f^{(j-1)}(x, u(x)) \\
&= \frac{\partial^j}{\partial x^j} f(x, u(x)) \\
&= \frac{\partial^j}{\partial x^j} u(x).
\end{align*}

In our algorithm we shall evaluate the known functions $f^{(j)}(x, v)$ at points $(x, v)$ of $D$ which are "close" to the points $(x, u(x))$ in order to obtain approximations $f^{(j)}(x, v) \approx d^{j+1} u(x)/dx^{j+1}$. Notice that the $f^{(j)}(x, v)$, $0 \leq j \leq m - 1$, are continuous in $D$ by our assumption that $f \in C'[D]$.

We shall denote $d^j y/dx^j$ by $y^{(j)}$ or $y^{(j)}(x)$, and if $y^{(j)}$ is continuous at $x_i$, we may denote its value there by $y^{(j)}_{i+}$. The notation $y^{(j)}_{i+}$ will be used for the right-hand limit of $y^{(j)}$ at $x_i$ when $y^{(j)}$ is discontinuous at $x_i$ and sometimes, for convenience, even when $y^{(j)}$ is continuous there.