Cusp Forms and the Index Theorem for Manifolds with Boundary

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1. Introduction

Let $G = SU(2,1)$ and let $D = G/K$ where $K$ is a maximal compact subgroup of $G$. Let $\Gamma$ be an arithmetic subgroup of $G$, and for simplicity, assume that $\Gamma$ acts freely on $D$. It was shown in [3] that the dimension of the space of cusp forms of weight $m$ on $D$ is given by

$$\frac{1}{2} (3m-1)(3m-1) T(\Gamma \backslash D) - \sum_{\text{cusps (indices omitted)}} \frac{1}{12} \beta,$$

where for each cusp, $\beta$ equals the number of linearly independent theta functions of a certain type and where $T(\Gamma \backslash D)$ is the Todd characteristic. This formula was derived in [3] by compactifying $\Gamma \backslash D$ and then applying the theorem of Riemann-Roch. The purpose of this paper is to explain how the above formula can be derived from the Atiyah-Patodi-Singer index theorem for manifolds with boundary [1] using the $\delta + \delta^*$ operator. The results of both this paper and [3] can be generalized to $SU(n,1)$ with no essential changes.

2. A Truncation of $\Gamma \backslash D$

Let $G = SU(2,1)$ and let $D = G/K$ where $K$ is a maximal compact subgroup of $G$. Let $\Gamma$ be an arithmetic subgroup of $G$, and for simplicity, assume that $\Gamma$ acts freely on $D$. Suppose $\Gamma$ has a cusp at infinity, that is, suppose $\Gamma \cap P$ is nontrivial, where $P$ is the isotropy subgroup of $G$. Then

$$D = \{ (z, u, 1) \in \mathbb{C}P(2) | 2 \text{Im} z - |u|^2 > 0 \}. $$

The group $G$ acts on $D \subset \mathbb{C}P(2)$ in the obvious way. Suppose $\Gamma$ is an arithmetic subgroup of $G$ that acts freely on $D$. Suppose, moreover, that $\Gamma$ has a cusp at infinity. That is, suppose $\Gamma \cap P$ is nontrivial, where $P$ is the isotropy subgroup of $G$.
corresponding to \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}P(2) \). The group \( P \) can be decomposed as \( P = NAM \), where

\[
A = \left\{ \begin{pmatrix} \delta & 0 \\ 1 & \delta^{-1} \end{pmatrix} \middle| \delta > 0 \right\}, \quad M = \left\{ \begin{pmatrix} \beta & \beta^{-2} \\ 0 & 1 \end{pmatrix} \middle| \beta^2 = 1 \right\},
\]

and

\[
N = \left\{ \begin{pmatrix} 1 & a & \frac{1}{3} \|a\|^2 + r \\ 0 & 1 & i\tilde{a} \\ 0 & 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C}, \quad r \in \mathbb{R} \right\}.
\]

Assume for simplicity that \( \Gamma \cap P = \Gamma \cap N \). Notice that if \( \alpha = 2 \text{Im} z - |u|^2 \), then \( \alpha \) is invariant under the action of \( N \) on \( D \). If \( T \) is sufficiently large, then the action of \( \Gamma \) on \( \{(z, u, 1) | T - 1 \leq z \leq T\} \) is the same as the action of \( \Gamma \cap N \). We can truncate \( \Gamma \setminus D \) near infinity by insisting that \( 0 < \alpha \leq T \). We can truncate \( \Gamma \setminus D \) near each of the other cusps in an analogous way. Notice that the boundary of our truncated manifold near infinity is given by \( \Gamma \cap N \setminus \{(z, u, 1) \in D | \alpha = T\} \).

3. An Invariant Metric on \( D \)

There is a naturally given invariant metric \( ds^2 \) on \( \Gamma \setminus D \) that can be obtained by computing \( \partial \bar{\partial} \log k(P, P) \), where \( k \) is (up to a constant) the Bergman kernel function of the domain \( D \), and where \( P = (z, u, 1) \in D \). It turns out that \( k(P, P) = \alpha^{-2} \) and that

\[
ds^2 = 3\alpha^{-2}(-idz - \tilde{u}du) \cdot (idz - \tilde{u}du) + 3\alpha^{-1}du \cdot d\tilde{u},
\]

where \( \alpha = 2 \text{Im} z - |u|^2 \). Since \( \alpha \) is invariant under the action of \( \Gamma \cap N \) near the cusp at infinity, it follows that we can "deform" \( ds^2 \) on the cusp at infinity to the slightly simpler metric:

\[
ds^2 = (-idz - \tilde{u}du) \cdot (idz - \tilde{u}du) + du \cdot d\tilde{u}.
\]

Now let

\[
\begin{pmatrix} dz' \\ du' \end{pmatrix} = \begin{pmatrix} -1 & -\tilde{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dz \\ du \end{pmatrix}.
\]

Then \( ds^2 = dz' \cdot d\bar{z}' + du' \cdot d\bar{u}' \), and \( \frac{\partial}{\partial z'}, \frac{\partial}{\partial u'} \) gives an orthonormal basis for the tangent bundle. Explicitly,

\[
\begin{pmatrix} \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial u'} \end{pmatrix} = \begin{pmatrix} i & 0 \\ i\tilde{u} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial u} \end{pmatrix}.
\]

We will use these orthonormal coordinates to compute \( \tilde{\partial}^* \) in the next section.