Radiative muon capture in light atoms

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Total cross sections for radiative muon capture into inner shells of light atoms are evaluated.

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One important problem in muon catalyzed fusion (MCF) \cite{1} is the determination of the muon capture rate into hydrogen or helium. This capture rate is an essential ingredient for the number of possible fusion cycles per muon. Muon capture can proceed either by Auger transitions, by three-body collisions incorporating neighbouring atoms or by radiative transitions. In this note we concentrate on the evaluation of the total cross section for the latter process which can be measured independently. After muon catalyzed fusion the muon can be ionized to continuum states with kinetic energies $E = \frac{1}{2} \mu v^2$ of about 10 keV and maximum energies up to the order of 100 keV \cite{2,3}. Thus we may restrict our calculations to this energy domain which also allows to stay within the nonrelativistic framework. The ionization rate of the bound muon after nuclear fusion may be calculated rather reliably thereby providing the probability distribution of the muon kinetic energy. Prior to our analytical evaluations radiative capture rates have been determined within a numerical treatment \cite{4,5} which, however, covers a more restricted energy range concerning the initial muon. In addition, it can be deduced from \cite{4,5} that the Auger process generally represents the dominant capture mode. The Auger capture of a muon from a continuum state into a bound state is accompanied by a simultaneous Coulomb excitation of a bound electron to a continuum state.

In first-order perturbation theory we can employ Fermi's golden rule to represent the radiative capture cross section from an initial continuum state $|\psi_i\rangle$ to a final bound state $|\psi_f\rangle$ with principal quantum number $n$ and orbital angular momentum quantum number $l$ as \cite{6-10}

\begin{equation}
\sigma_{nl} = \frac{V}{v} \sum_{x=1,2} \frac{\alpha}{2\pi \mu^2 c^2} \int |\langle \psi_f | e^{-iQ \cdot \hat{p}} | \psi_i \rangle|^2 d\Omega_Q \tag{1}
\end{equation}

$\hat{p}$ with $x=1,2$ denote the two orthogonal polarization vectors of the emitted photon which are perpendicular to the photon momentum $Q$, i.e. $\hat{p} \cdot Q = 0$. $\hat{p}$ is the momentum operator acting on the initial continuum state $|\psi_i\rangle$. The direction of the impinging muon with momentum $\hbar q = \mu v$ is chosen to be the quantization axis ($z$-axis); $v = |v|$. Specific recoil contributions which are not compensated by taking the ordinary reduced mass $\mu$ of the muon will be neglected. Since we are basically dealing with capture into inner shells of light atoms with spatial expectation values of the muon being much smaller than the first electron Bohr orbit we also may neglect electron screening as well as nuclear size corrections. Of course, the theory of radiative recombination is well known. However, for the sake of completeness we cite the basic ingredients of our calculations.

In (1) an integration over the angular domain $d\Omega_Q$ of the emitted photon has to be performed. The photon energy $\hbar \omega$ is determined by $\hbar \omega = E + E_{nl}$ with the absolute value of the binding energy of the final bound state $E_{nl} = \frac{1}{2} (Z \alpha)^2 \mu c^2/n^2$. The factor $V$ in front of (1) is totally compensated by the normalization of the initial state $|\psi_i\rangle$ being normalized to one particle per unit volume $V$. As the most trivial assumption
\( \Psi_l \) can be described by a plane wave \( \Psi_l = e^{i q \cdot r} \sqrt{V} \) which corresponds to the first Born approximation.

To account for Coulomb corrections we take for \( \Psi_l \) a sum over partial waves with angular momentum quantum numbers \( l \) (\( q = |q| \))

\[
\Psi_l = \frac{1}{\sqrt{V}} \sum_{n=0}^{\infty} (2l+1)^{1/2} e^{i q \cdot r} F_l(-\eta; q r) P_l(\cos \theta)
\]  

with the phase shift \( \delta_l = \arg F(l+1-i\eta) \) and the Sommerfeld parameter \( \eta = Z \alpha c / v \). \( F_l(-\eta; q r) \) denotes the regular Coulomb wavefunction with the asymptotic behaviour \( F_l(-\eta; q r \to \infty) = \sin(q r + \eta \ln(2q r) - \frac{1}{2} \pi + \delta_l) \). The final bound state wavefunction reads

\[
\Psi = \Psi_{l m}(r) = R_{n l}(r) Y_{l m}(\theta, \phi).
\]  

For hydrogen-like systems we utilize the analytical expression [8]

\[
R_{n l} = \frac{1}{r} \left[ \frac{n-l-1}{\eta^2} \left( \frac{2}{(n+l)!} \right)^{3/2} a \right]^{l+1} \sin^{(l+1)n} a \frac{L_{2l+1}^{2l+1}(\frac{2r}{n a})}{(\sqrt{\eta} a)}
\]  

with the muon Bohr radius \( a = a_0 / Z \) and \( a_0 = h c / (\mu c^2) \).

The generalized Laguerre polynomials may be computed according to

\[
L_{2l+1}^{2l+1}(x) = (-1)^{2l+1} \frac{(n+l)!^2}{(2l+1)!(n-l-1)!} F_l(l+1-n, 2l+2; x).
\]  

If we restrict our considerations to the atomic K-shell this reduces to \( \Psi_{l=1} = (\pi a^2)^{-1/4} e^{-r/a_0} \).

In plane-wave Born approximation one readily obtains the total cross section for radiative muon capture into the 1s-state

\[
\sigma_{1s} = 64 \pi a^2 \frac{h \omega}{v} \frac{\sin^3 \theta}{(1 + a^2 Q^2 + a^2 Q^2) - 2 a Q \cos \theta} \ d \theta.
\]  

with \( q = \mu v / h \) and \( Q = \omega / c \). In the long-wavelength approximation \( (Q a \ll 1) \) which is totally justified in the considered energy domain this can be further simplified to

\[
\sigma_{1s} = \frac{256}{3} \pi a^3 \frac{(h \omega)^3}{v (h \omega - E_{1s})} \frac{E_{1s}^2}{2 \pi a_0^2}.
\]  

Using the partial-wave representation (2) and the long-wavelength approximation \( \exp \{-i Q \cdot r \} \approx 1 \) we can perform the transformation

\[
\langle \Psi_f | \hat{p} | \Psi_i \rangle = i \mu \omega \langle \Psi_f | \hat{R} | \Psi_i \rangle
\]  

to dipole matrix elements. Equation (8) is a consequence of the commutator relation \( [\mathbf{p}, \hat{H}_0] = i \hbar \) and of the fact the \( \Psi_f \) and \( \Psi_i \) are eigenstates of the Hamiltonian \( \hat{H}_0 \). This leads to

\[
\sigma_{nl} = \frac{4}{3} \pi \frac{(h \omega)^3}{v q} V \langle \Psi_f | R | \Psi_i \rangle^2.
\]  

After angular integration [7, 9] it results for the muon capture into a bound state with the quantum numbers \( n \) and \( l \)

\[
\sigma_{nl} = \frac{4}{3} \pi \alpha c \left( \frac{h \omega}{v} \right)^3 \int \left[ \int_0^{\infty} R_{n l} F_{l+1}(-\eta; q r) r^2 \ dr \right]^2
\]

\[
+ (l+1) \left( \int_0^{\infty} R_{n l} F_{l+1}(-\eta; q r) r^2 \ dr \right)^2 \right] \right)
\]  

The cross section (10) agrees with the equivalent result for the radiative transition rate of Haff and Tombrello [5]. Restricting again ourselves to the 1s-state one can perform the radial integrals in (10) analytically. The capture cross section into the 1s-state finally reads

\[
\sigma_{1s} = \frac{256}{3} \pi a^3 \frac{E_{1s}^3}{(h \omega)^2 (h \omega - E_{1s})} \frac{\exp \{-4 \eta \arctan(1/\eta)\}}{1 - \exp \{-2 \pi \eta \}} \ a_0^2.
\]  

The total cross section for radiative recombination of electrons [7, 9] follows simply by replacing the reduced mass \( \mu \) of the muon by the corresponding value for the electron. In the limit \( \eta \to 0 \) and for kinetic energies \( E \gg E_{1s} \) of the muon we obtain from (11)

\[
\sigma_{1s} = \frac{128}{3} \pi a^3 \frac{E_{1s}}{h \omega (h \omega - E_{1s})} \frac{E_{1s}^2}{(h \omega)^2} \ a_0^2.
\]  

(12) coincides with the result (7) of the Born approximation in the long-wavelength limit.

Using the computer code [11] for the Coulomb wavefunction and the representation (4) of the radial bound state wavefunction we also performed the radial integrals in (10) numerically with high-precision Gaussian quadrature procedures [12]. Complete agreement with the analytical result (11) was achieved. There are two major motivations for this numerical treatment. Firstly, it provides a simple test of the analytical integration and its numerical evaluation, and