TWO SUFFICIENT CONDITIONS FOR THE UNIVALENCE
OF ANALYTIC FUNCTIONS

V. P. Mikka

In this article we obtain sufficient conditions for the univalence of \( n \)-symmetric analytic functions in the region \(|z| \geq 1\) and in the disk \(|z| \leq 1\). We examine the question of univalent variation of functions analytic in \(|z| < 1\) and mapping \(|z|=1\) onto a contour with two zero angles. We give an application of these results to the fundamental converse boundary-value problems.

It is known (see, e.g., [1, p. 28]) that in the class of functions

\[
F(z) = a_0 z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad |z| > 1,
\]

Zhukov's function \( F(z) = \frac{1}{z} (z + 1/z) \) is an example of a convex function which is multivalent in \(|z| \geq 1\). Therefore, an arbitrary extension of the class of convex functions will yield even more multivalent functions. In this article we shall offer a univalent extension of the class of convex functions.

We will also establish a method of extending Kaplan's class [2] in the disk \(|z| \leq 1\). In particular, we will distinguish the regular functions, univalent in \(|z| \leq 1\), which map \(|z| = 1\) onto a contour consisting of two arcs of concentric circles. This central result, proved for functions \( F(z) \) in the region \(|z| > 1\) and for the extension of Kaplan's class in \(|z| \leq 1\), is applied to the question of the univalence of solutions of the converse boundary-value problems [3].

§ 1. Sufficient Conditions for the Univalence
of \( n \)-Symmetric Functions

THEOREM 1. Let the function

\[
F_n(z) = a_1 z + \frac{a_{n-1}}{z} + \frac{a_{2n-1}}{z^{2n-1}} + \cdots, \quad n \geq 2,
\]

be \( n \)-symmetric and regular in \( 1 < |z| < \infty \), and have a continuous derivative \( F_n'(z) \neq 0 \) in \(|z| \geq 1\). Then \( F_n(z) \) is univalent in \(|z| \geq 1\) if one of the following two conditions is satisfied:

\[
\begin{align}
\int_{\gamma} d\gamma_F_n(1, \theta) & \leq -2\pi/n, & \int_{L^+} d\gamma_{F_n}(1, \theta) & = -2\pi/n; \\
\int_{\gamma} d\gamma_{F_n}(1, \theta) & \geq -\pi, & \int_{L^-} d\gamma_{F_n}(1, \theta) & = -2\pi/n,
\end{align}
\]

where \( L^- \) is an arbitrary arc of length \( 2\pi/n \) of the circle \(|z| = 1\) in a clockwise direction, and \( \gamma^- \) is an arbitrary connected part of \( L^- \); \( \gamma_{F_n}(1, \theta) \) is the angle between the real axis and the tangent to the image of \(|z| = 1\) at the point \( z = e^{1\theta} \), under the mapping of \(|z| \geq 1\) by the function \( F_n(z) \).
**Proof.** The image of $|ξ| ≥ 1$ does not contain any branch points, for otherwise the condition

$$\int_{|ξ|=1} dΓ_{F_n}(1, θ) = -2π$$

will not be fulfilled. Suppose that the function $z = F_n(ξ) ≠ 0$ is $p$-sheeted in $|ξ| ≥ 1$ for $p ≥ 2$. Denote by $Δ_n$ the image of $|ξ| > 1$ under the mapping by the function $F_n(ξ)$. Then $Δ_n$ may be considered as a one-sheeted region on the no less than $p$-sheeted Riemann surface $Σ$ over the $z$ plane. In view of this we may assume that $Σ$ possesses the following property: If $z = z_0$ is a branch point, then the points $z_k = z_0 e^{i2πk/p}$, $k = 1, \ldots, \ n - 1$, are also branch points of $Σ$. The region $Δ_n$ can be many-sheeted only if $Δ_n$ encloses at least one branch point of $Σ$, for instance $z_0$. Denote by $Γ_0$ the boundary enclosing the part of $Δ_n$ which lies on the sheet of $Σ$ containing $z = ∞$. Let $z_1^*$ and $z_2^*$ be those points of $Γ_0$ which project onto the one point $z_1$. Since the function $F_n(ξ)$ is single-valued in $|ξ| > 1$ and is continuous up to the boundary, $F_n(ξ)$ is single-valued in $|ξ| ≥ 1$. Let $ξ_1 = e^{iθ_1}$ and $ξ_2 = e^{iθ_2}$ be such that

$$z_1^* = F_n(ξ_1) = F_n(ξ_2) = z_2^*.$$  

(1.4)

Then $Γ_0$ is the image of the arc $l_0$ with ends at the points $e^{iθ_1}$ and $e^{iθ_2}$. We will show that $θ_1 - θ_2 < 2π/n$.

By (1.1) we have

$$F_n(z_0 e^{-i2πk/p}) = e^{-i2πk/p} F_n(z_0),$$  

(1.5)

and therefore $ξ_j e^{-i2πk/p}$, $j = 1, 2; k = 0, \ldots, \ n - 1$, satisfy the equations $z_j e^{-i2πk/p} = F_n(ξ_j)$, $j = 1, 2; k = 0, \ldots, \ n - 1$, where the arcs $Γ_k$ are obtained by rotating the arc $l_0$ through the angle $-2πk/n$ about $z = 0$, and are the images of the arcs $l_k$ with ends at the points $ξ_je^{-i2πk/p}$, $ξ_je^{-i2πk/p}$. We see that the arcs $l_k$, $k = 0, \ldots, \ n - 1$, do not have any points in common. Suppose the converse is true. Let $l_k^*$ have a part in common with $l_0^*$, which we denote by $l_{0k}$. If $l_{0k}$ coincides with the point $ξ_1$, we obtain $z_{1k} = z_{1} e^{-i2πk/p} = z_{1k} e^{-i2πk/p}$, which cannot hold for $0 < k < n - 1$, since $z_{1k} = z_{2k}^* ≠ 0$. Denote the image of $l_{0k}$ by $Γ_{0k}$.

It is evident that the beginning of $Γ_{0k}$ is the point $z_{1k}^*$, and the end is the point $z_{1k}^* e^{-i2πk/p} = z_{1k} e^{-i2πk/p}$. Since $l_{0k} ⊂ l_k$, we have $Γ_{0k} ⊂ Γ_k$; and since $l_{0k} ⊂ l_k$, then $Γ_{0k} ⊂ Γ_k$, by virtue of the fact that $F_n(ξ)$ is single-valued in $|ξ| ≥ 1$. Thus we see that $Γ_k$ lies on the same sheet of $Σ$ as $Γ_0$; but this last condition contradicts the fact that $Δ_n$ may be regarded as one-sheeted on the surface $Σ$, since we have shown that $Γ_{0k}$ lies on one sheet of $Σ$. We have arrived at a contradiction, and therefore the arcs $l_k$, $k = 0, \ldots, \ n - 1$, do not have any points in common, i.e., their length is less than $2π/n$. By our assumption the contour $Γ_0$ encloses the point $z_0$, and therefore the contour $Γ_k$ must enclose the point $z_k$ for $k = 1, \ldots, \ n - 1$.

A variation in the angle of the tangent to $Γ_k$ satisfies the inequality

$$\int_{Γ_k} dΓ_{F_n}(1, θ) < -π,$$

which is equivalent to the proved fact

$$\int_{Γ_k} dΓ_{F_n}(1, θ) < -π,$$

but this contradicts condition (1.3).

Conditions (1.2) and (1.3) are equivalent, because we have the equation

$$\int_{Γ_k} dΓ_{F_n}(1, θ) = -2π/n - \int_{Γ_k} dΓ_{F_n}(1, θ).$$

As a consequence of the above proof we obtain the following property of the function $F_n(ξ)$.

**THEOREM 2.** The $n$-symmetric function $F_n(ξ) ≠ 0$ with derivative $F_n(ξ) ≠ 0$ in $|ξ| ≥ 1$ is univalent if and only if it is univalent in every sector with an angle of $2π/n$.

We now show that $F_n(ξ) ≠ 0$ in the region $|ξ| ≥ 1$ if inequality (1.3) is satisfied. Suppose the converse is true. Let $ξ_0 ∈ |ξ| ≥ 1$ be the point having greatest modulus such that $F_n(ξ_0) = 0$. By Eq. (1.5) the points