In this article we obtain sufficient conditions for the univalence of \( n \)-symmetric analytic functions in the region \( |\xi| \geq 1 \) and in the disk \( |\xi| \leq 1 \). We examine the question of univalent variation of functions analytic in \( |\xi| < 1 \) and mapping \( |\xi|=1 \) onto a contour with two zero angles. We give an application of these results to the fundamental converse boundary-value problems.

It is known (see, e.g., [1, p. 28]) that in the class of functions

\[
F(\xi) = a_n \xi^n + \frac{a_{n-1}}{\xi^{n-1}} + \cdots + \frac{a_k}{\xi^{k-n}} + \cdots, \quad |\xi| > 1,
\]

Zhukov's function \( F(\zeta) = \frac{1}{2} (\zeta + 1/\zeta) \) is an example of a convex function which is multivalent in \( |\zeta| \geq 1 \). Therefore, an arbitrary extension of the class of convex functions will yield even more multivalent functions. In this article we shall offer a univalent extension of the class of convex functions.

We will also establish a method of extending Kaplan's class [2] in the disk \( |\zeta| \leq 1 \). In particular, we will distinguish the regular functions, univalent in \( |\zeta| \leq 1 \), which map \( |\zeta|=1 \) onto contours consisting of two arcs of concentric circles. This central result, proved for functions \( F(\zeta) \) in the region \( |\zeta| > 1 \) and for the extension of Kaplan's class in \( |\zeta| \leq 1 \), is applied to the question of the univalence of solutions of the converse boundary-value problems [3].

§ 1. Sufficient Conditions for the Univalence of \( n \)-Symmetric Functions

THEOREM 1. Let the function

\[
F_n(\xi) = a_{n-1} \xi + \frac{a_{n-1}}{\xi^{n-1}} + \cdots + \frac{a_k}{\xi^{k-n}} + \cdots, \quad n \geq 2,
\]

be \( n \)-symmetric and regular in \( 1 < |\xi| < \infty \), and have a continuous derivative \( F'_n(\xi) \neq 0 \) in \( |\xi| \geq 1 \). Then \( F_n(\xi) \) is univalent in \( |\xi| \geq 1 \) if one of the following conditions is satisfied:

\[
\begin{align*}
\int_{\gamma^-} d\gamma_{F_n}(1, \theta) &= \pi - 2\pi/n, \\
\int_{L^-} d\gamma_{F_n}(1, \theta) &= 2\pi/n;
\end{align*}
\]

\[(1.2)\]

\[
\begin{align*}
\int_{\gamma^-} d\gamma_{F_n}(1, \theta) &= -\pi, \\
\int_{L^-} d\gamma_{F_n}(1, \theta) &= -2\pi/n,
\end{align*}
\]

\[(1.3)\]

where \( L^- \) is an arbitrary arc of length \( 2\pi/n \) of the circle \( |\xi| = 1 \) in a clockwise direction, and \( \gamma^- \) is an arbitrary connected part of \( L^- \); \( \gamma_{F_n}(1, \theta) \) is the angle between the real axis and the tangent to the image of \( |\xi|=1 \) at the point \( \zeta = e^{1/\theta} \), under the mapping of \( |\xi| \geq 1 \) by the function \( F_n(\xi) \).

Proof. The image of \(|\xi| \geq 1\) does not contain any branch points, for otherwise the condition

$$\int_{|\xi|=1} d\Gamma_{\xi} F_n(1, \theta) = -2\pi$$

will not be fulfilled. Suppose that the function \(z = F_n(\xi) \neq 0\) is \(p\)-sheeted in \(|\xi| \geq 1\) for \(p \geq 2\). Denote by \(\Delta_n\) the image of \(|\xi| > 1\) under the mapping by the function \(F_n(\xi)\). Then \(\Delta_n\) may be considered as a one-sheeted region on the no less than \(p\)-sheeted Riemann surface \(\Sigma\) over the \(z\) plane. In view of this we may assume that \(\Sigma\) possesses the following property: If \(z = z_0\) is a branch point, then the points \(z_k = z_0 e^{-i2\pi k/n}, k = 1, \ldots, n-1\), are also branch points of \(\Sigma\). The region \(\Delta_n\) can be many-sheeted only if \(\Delta_n\) encloses at least one branch point of \(\Sigma\), for instance \(z_0\). Denote by \(\Gamma_0\) the boundary enclosing the part of \(\Delta_n\) which lies on the sheet of \(\Sigma\) containing \(z = \infty\). Let \(z_1^*\) and \(z_2^*\) be those points of \(\Gamma_0\) which project onto the one point \(z_1\). Since the function \(F_n(\xi)\) is single-valued in \(|\xi| > 1\) and is continuous up to the boundary, \(F_n(\xi)\) is single-valued in the region \(|\xi| > 1\). Let \(\xi_1 = e^{i\theta_1}\) and \(\xi_2 = e^{i\theta_2}\) be such that

$$z_1^* = F_n(\xi_1) = F_n(\xi_2) = z_1^*.$$

Then \(\Gamma_0\) is the image of the arc \(\xi_0\) with ends at the points \(e^{i\theta_1}\) and \(e^{i\theta_2}\). We will show that \(\theta_1 - \theta_2 < 2\pi/n\).

By (1.1) we have

$$F_n(\xi e^{-i2\pi k/n}) = e^{-i2\pi k/n} F_n(\xi).$$

and therefore \(z_k e^{-i2\pi k/n}, j = 1, 2; k = 0, \ldots, n-1\), satisfy the equations \(z_j e^{-i2\pi k/n} = F_n(\xi), j = 1, 2; k = 0, \ldots, n-1\), where the arcs \(\Gamma_k\) are obtained by rotating the arc \(\Gamma_0\) through the angle \(-2\pi/k/n\) about \(z = 0\), and are the images of the arcs \(\Gamma_0\) with ends at the points \(z_0 e^{-i2\pi k/n}, z_0 e^{-i2\pi k/n}\). We see that the arcs \(\Gamma_k, k = 0, \ldots, n-1\), do not have any points in common. Suppose the converse is true. Let \(\Gamma_k\) have a part in common with \(\Gamma_k\), which we denote by \(\Gamma_k\). If \(\Gamma_k\) coincides with the point \(z_1\), we obtain \(z_k = z_1 e^{-i2\pi k/n} = z_1 e^{-i2\pi k/n}\), which cannot hold for \(0 < k < n-1\), since \(z_1 e^{-i2\pi k/n} \neq z_k\). Denote the image of \(\Gamma_k\) by \(\Gamma_k\).

It is evident that the beginning of \(\Gamma_k\) is the point \(z_1^*\), and the end is the point \(z_1^* = z_1^* = z_1^*\). Since \(\Gamma_k \subset \Gamma_0\), we have \(\Gamma_k \subset \Gamma_0\); and since \(\Gamma_k \subset \Gamma_0\), then \(\Gamma_k \subset \Gamma_0\), by virtue of the fact that \(F_n(\xi)\) is single-valued in \(|\xi| \geq 1\). Thus we see that \(\Gamma_k\) lies on the same sheet of \(\Sigma\) as \(\Gamma_0\); but this last condition contradicts the fact that \(\Delta_n\) may be regarded as one-sheeted on the surface \(\Sigma\), since we have shown that \(\Gamma_k\) lies on one sheet of \(\Sigma\). We have arrived at a contradiction, and therefore the arcs \(\Gamma_k, k = 0, \ldots, n-1\), do not have any points in common, i.e., their length is less than \(2\pi/n\). By our assumption the contour \(\Gamma_0\) encloses the point \(z_0\), and therefore the contour \(\Gamma_k\) must enclose the point \(z_k\) for \(k = 1, \ldots, n-1\).

A variation in the angle of the tangent to \(\Gamma_k\) satisfies the inequality

$$\int_{\Gamma_k} d\tau_{\Gamma_k} F_n(1, \theta) < -\pi,$$

which is equivalent to the proved fact

$$\int_{\Gamma_k} d\tau_{\Gamma_k} F_n(1, \theta) < -\pi,$$

but this contradicts condition (1.3).

Conditions (1.2) and (1.3) are equivalent, because we have the equation

$$\int_{\Gamma_0} d\tau_{\Gamma_0} F_n(1, \theta) = -2\pi/n - \int_{\Gamma_k} d\tau_{\Gamma_k}(1, \theta).$$

As a consequence of the above proof we obtain the following property of the function \(F_n(\xi)\).

**Theorem 2.** The \(n\)-symmetric function \(F_n(\xi) \neq 0\) with derivative \(F_n(\xi) \neq 0\) in \(|\xi| \geq 1\) is univalent if and only if it is univalent in every sector with an angle of \(2\pi/n\).

We now show that \(F_n(\xi) \neq 0\) in the region \(|\xi| \geq 1\) if inequality (1.3) is satisfied. Suppose the converse is true. Let \(z_0 \in |\xi| \geq 1\) be the point having greatest modulus such that \(F_n(z_0) = 0\). By Eq. (1.5) the points