ABELIAN VARIETIES IN CHARACTERISTIC p

Yu. G. Zarkhin

UDC 512

In this paper Tate's finiteness conjecture for isogenies of polarized Abelian varieties in characteristic $p > 2$ is proved. From this conjecture it is deduced that Tate modules are semisimple and that Tate's conjecture on the homomorphisms of Abelian varieties is valid.

Introduction. Let $p$ be a prime number $\neq 2$, $E$ a field of finite type over a finite field $k$ of characteristic $p$, $G = \text{gal} (\bar{E}/E)$, and $l$ a prime number $\neq p$.

THEOREM. If $A$ is an Abelian variety over $E$ and $d$ is a natural number such that $(d, p) = 1$, then there exists only a finite number of pairs $(B, L)$, up to an $E$-isomorphism, consisting of an Abelian variety $B$ over $E$ and an $L \in \text{Pic} B$ such that:

1) $L$ is an ample sheaf of degree $d$;
2) there exists an isogeny $\varphi : B \rightarrow A$ whose degree is prime to $p$.

COROLLARY 1. $V_l(A)$ is a semisimple $G$-module.

COROLLARY 2. Tate's conjecture on homomorphisms is valid for Abelian varieties over $E$.

COROLLARY 3. Let $X$ be a product of curves and Abelian varieties over $E$, $X = X \otimes E$. Then the canonical mapping

$$NS(X) \otimes \mathbb{Q} \rightarrow H^1(X)(1)$$

is bijective.

The proof of Corollaries 1-3 may be found in [1, 2].

Remark. For fields of transcendence degree 1 over a finite field, the theorem was proved in [3]. In the general case it was deduced from conjectures on the resolution of singularities not proved in characteristic $p$ [4].

Throughout this paper we shall use the ideas, results, and notation in [3, 4].

§ 1. Method of Proof. 1.0. Let $F$ be a field of characteristic $\neq 2$; $\Omega$ the set of all valuations $v$ on $\bar{F}$ trivial on the prime subfield; $X$ and $Y$ Abelian varieties over $F$ separably isogeneous to each other; $L$ and $M$ very ample, totally symmetric, invertible sheaves on $X$ and $Y$, respectively. We choose totally symmetric $\theta$-structures on $(X, L)$ and $(Y, M)$. Points of the projective spaces — theta constants — correspond to them:

$$(g_L(\alpha))_{\alpha \in \Omega(X)} \subseteq \mathbb{P} |V(\delta_L)|(\bar{F})$$

sand

$$(g_M(\alpha))_{\alpha \in \Omega(Y)} \subseteq \mathbb{P} |V(\delta_M)|(\bar{F}).$$

(The precise definitions may be found in [3].)

*The author is using the possibility of correcting an error in the formulation of Lemma 4.5 in [4, p. 467]. The conclusion of the lemma should read as follows: "Then $G \in \text{Pic} T."$
1.1. LEMMA. One can choose collections of homogeneous coordinates
$$\{q_L(a)\}_{a \in K(\delta_L)} \subseteq V(\delta_L)(\bar{F}_S) \cup \{q_M(a)\}_{a \in K(\delta_M)} \subseteq V(\delta_M)(\bar{F}_S)$$
so that
$$\min_{a \in K(\delta_L)} v(q_L(a)) = \min_{a \in K(\delta_M)} v(q_M(a))$$
for $v \in \Omega$.

1.2. COROLLARY. Lemma 1.1 remains valid if one considers any set of pairwise isogeneous varieties or, equivalently, all Abelian varieties separably isogeneous to a given variety $X$.

Lemma 1.1 will be proved in § 2.

1.3. Let $E$ be an algebraic function field over a finite constant field $k$ of characteristic $p \neq 2$; $X$ an Abelian variety over $E$; and $d$ a natural number, $(d \text{ char } E) = 1$. Then there exists a finite separable extension $E'$ of $E$ satisfying the following condition: If $Y$ is an Abelian variety over $E$ isogeneous to $X$, then
$$q_L Y \subseteq Y(E').$$
This is a special case of Lemma 4.7 in [4].

1.4. LEMMA. Under the assumption and notation of Lemma 1.3, there exists a natural number $n$ such that for all pairs $(Y, L)$, where $L$ is a very ample, totally symmetric invertible sheaf of degree $d$, one can choose collections of homogeneous coordinates $\{q_L(a)\}_{a \in K(\delta_L)} \subseteq V(\delta_L)(E')$ so that:

1) Corollary 1.2 holds;
2) for all $a_1, \ldots, a_n \in K(\delta_L)$
$$\prod_{i=1}^n q_L(a_i) \subseteq E'.$$

1.5. Remark. The condition of the lemma guarantees that the appropriate point of projective space is $E'$-rational, i.e.,
$$\{q_L(a)\}_{a \in K(\delta_L)} \subseteq P(V(\delta_L)(E')).$$

Lemma 1.4 will be proved in § 3.

1.6. Deduction of the Finiteness Conjecture from Lemma 1.4. Replacing $k$ by a finite extension $k'$ where necessary, consider a projective normal model $S$ of $E$ over $k'$.

Then all of $q_L(a_1), q_L(a_2), \ldots, q_L(a_n)$ belong to one linear system, which is finite-dimensional (see [5, Vol. 2, Appendix 4]) and finite (everything is taking place over a finite field $k'$).

It follows that the set of all
$$\prod_{i=1}^n q_L(a_i)$$
is finite and, therefore, that the set of all $\{q_L(a)\}_{a \in K(\delta_L)}$, which uniquely define $(Y, L)$, is finite.

It remains to use the fact that $H^1(\text{gal } (E'/E), \text{Aut } (X, L))$ is finite. This proves the finiteness conjecture.

§ 2. Proof of Lemma 1.2. 2.0. All the arguments in this section were actually made in [3] (see also [4, § 3]).

We say that $(X, L)$ and $(Y, M)$ are $\Omega$ equivalent if they satisfy Lemma 1.2: $(X, Y) \sim (Y, M)$. $\Omega$ equivalence does not depend on the choice of a $\theta$ structure (a change of $\theta$ structure implies a linear transformation of "theta constants" with "constant" coefficients in the closure of the prime subfield).