Algebraic Methods for the Calculation of Radiation Exchange in an Enclosure

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Abstract. The algebraic methods of Hottel and Sarofim, Gebhart and one recently published by the authors for the calculation of radiant interchange in an enclosure are examined. Each of these formulations, while written in a different form, is shown to be mathematically equivalent to the others. This equivalence is established by deriving the formulations of Hottel and Sarofim and Gebhart from that of the authors. Because of the analytical complexity of these forms their mathematical equivalence cannot be demonstrated by explicit methods. For this reason proof of equivalence is established by numerical programming using a digital computer. Three different numerical examples are used for this purpose. In addition, the mathematical structure underlying the absorption factor $B_{ij}$ of Gebhart and the total exchange factor $S_i S_j$ and total view factor $F_{ij}$ of Hottel and Sarofim is developed.

Nomenclature

$A$ area, [ft$^2$]
$A_{ij}$ cofactor of matrix $[a_{ij}]$, Eq. (15), [1]
$[a_{ij}]$ geometric view factor–reflectance matrix, Eq. (8)
$B_{ij}$ Gebhart absorption factor, Eq. (1), [1]
$D$ determinant of $[a_{ij}]$ matrix, [1]
DERGEBHOT name of authors’ computer program to calculate Gebhart’s $B_{ij}$ and Hottel’s $S_i S_j = A_{ij} F_{ij}$
$F_{ij}$ geometric view factor, [1]
$[F_{ij}]$ geometric view factor matrix, Eq. (9), [1]
$F_{ij}$ Gebhart’s script-F or total view factor, Eq. (32), [1]
$H$ incident radiation flux, Eq. (3), [Btu/hr-ft$^2$]
$I$ identity matrix, Eq. (10), [1]
$J$ radiosity flux, Eqs. (14), (16), (17), [Btu/hr-ft$^2$]
$K$ ($e/\rho \sigma T^4$, Eq. (13), [Btu/hr-ft$^2$]
$n$ number of surfaces in enclosure, [1]
$q_{ij}$ net radiation heat transfer at surface $A_j$, [Btu/hr]
$q_{ij}$ radiant heat transfer rate between surfaces $A_i$ and $A_j$, Eq. (31) [Btu/hr]
$q_{ij}$ energy absorbed at surface $A_j$ originating as emission at surface $A_i$, [Btu/hr]
$\varphi_{ij}$ net radiation heat transfer at surface $A_j$, [Btu/hr]
RADTQO name of authors’ computer program to use with formulation in [12]
$T$ temperature, [°F]
$x$ coordinate

Greek

$\alpha$ absorptance or absorptivity, [1]
$\beta$ angle, [1]
$[a_{ij}]$ matrix, Eq. (25), [1]
$\gamma_{ij}$ coefficients of the inverted matrix $[a_{ij}]^{-1}$, Eq. (15), [1]
$\epsilon$ emittance or emissivity, [1]
$\rho$ reflectance or reflectivity, [1]
$\sigma$ Stefan-Boltzmann constant

Subscripts

$i$ designation of row number in matrix; also $i$th surface
$j$ designation of column number in matrix; also $j$th surface

1. Introduction

The exchange of thermal radiation between the surfaces of an enclosure having arbitrary geometry is an important engineering design problem of current interest. Frequently, the information desired in such calculations is either the temperature of the radiating surfaces for a prescribed heat flux rate or the net rate of heat transfer at the surface for specified surface temperatures. In some instances, an enclosure may have heat flux rates fixed for some of the surfaces and temperatures given on others. The calculation of each of these quantities can, furthermore, be determined by the manner in which the enclosure interacts thermally with its external environment. This interaction can take many forms, including both convective and...
radiative exchange with an environment, a fixed external heat flux, an enthalpy transfer by the agency of fluid flow in ducts within the surface or heat generation in the surface. In terms of enclosure analysis these interactive effects represent boundary conditions or constraints imposed on the system of equations governing the radiation exchange within the enclosure. There are also other constraints imposed by the laws of thermodynamics which while not usually explicitly stated, represent immutable limits on both the physical system and the set of equations chosen to describe the system. For a steady-state condition the first law requires the sum of the net internal radiant heat transfer rates at the surfaces to be zero; the second law requires the specification of at least one temperature in the system, either at a surface or in the environment with which the system interacts.

For many engineering design situations the assumption of gray, diffuse radiating surfaces without the presence of participating media is satisfactory. The design calculations in such systems are generally carried out using an algebraic or numerical formulation with subsequent reduction of the equations on a digital computer or, in simple systems, by hand calculations. These kinds of formulations which assume uniform radiosity flux and uniform properties for each surface are numerical approximations to the more exact analytical formulations involving the solution of a system of integral equations. However, the difficulties encountered in the solution of such equations for an enclosure of arbitrary geometry are so enormous as to render any general attempt impractical for design purposes. On the other hand, the various algebraic methods may be relatively easily applied to very complex systems providing results having excellent correspondence with the more exact formulations. This is particularly true when a large memory digital computer is available for programming and solving the governing system of equations. Furthermore, the condition of variable surface radiosity flux, temperature and surface properties can be approximated to almost any degree by subdividing a surface into sufficiently small sub-sections. The penalty for this improvement is an increase in the magnitude and cost of the calculations.

In recent years several algebraic methods for radiation enclosure analysis and calculation have been published. Probably the earliest of these is that of Hottel [1]. Later this work was expanded by Hottel and Sarofim [2]. The basis for their method is the concept of the radiosity heat flux and, hence, is sometimes referred to as a radiosity method. Gebhart [3, 4] has approached the problem of enclosure analysis in a different way. His method is based on the definition of an absorption factor, the fraction of energy absorbed at a surface which originates as emission at another surface. Other work dealing with radiant exchange between the surfaces of an enclosure include those of Sparrow [5], Sparrow and Cess [6], Oppenheim [7], Eckert and Drake [8], Love [9], Wiebelf [10] and Siegel and Howell [11].

Recently the authors have published a paper on this problem [12] which treats an enclosure having gray, diffuse surfaces that are adiabatic or of known temperature. The formulation of this method is based on the concept of radiosity heat flux and is similar to that of Hottel [1, 2] although formulated in somewhat different manner. The emphasis is on engineering design application with the governing equations written for convenience in computing programing. The analysis is coupled with the external thermal interactions of the enclosure with its environment. A number of examples of the use of the method to enclosures are given and a comparison is made with the results of Hottel [1, 2] and Hottel and Sarofim [2] and Gebhart [3].

The various algebraic methods for enclosure analysis are mathematically identical. Although formulated in different ways their differences represent only the varying mathematical and physical perspectives of their authors. For a given set of conditions, each method produces results identical to that of the others. Usually certain types of calculations are more conveniently made with one method, rather than another, owing to the manner of its formulation. It is also possible to derive the form of any of the methods using another as a starting point.

The purpose of the present paper is to demonstrate the equivalence of the algebraic methods of Hottel and Sarofim [1, 2], Gebhart [3] and that given by the authors [12]. The equivalence will be established by using the enclosure analysis in [12] to derive the methods of [1, 2 and 3]. Specific numerical examples will then be given in which numerical results using each method are compared. These results will also be compared with such independent calculations made by others as are available. All numerical calculations made by the present authors were performed using the IBM 360/67 Digital Computer of The University of Michigan Computing Center.

In addition to showing equivalence of the various methods for enclosure calculations, a fundamental derivation of the elements which form the Gebhart absorption factor, \( B_{ij} \), the Hottel total-view factor, \( \Phi_{ij} \), and the total-exchange area, \( S_i S_j \), of Hottel and Sarofim will be given. All computer calculations have been performed using the author's program named DERGEBHOT.

2. Derivation of Gebhart's Absorption Factors

An enclosure of \( n \)-surfaces having arbitrary geometry is shown in Fig. 1. It will be assumed that all surfaces have gray, diffuse radiation characteristics. The consequences and applicability of this assumption are well documented [2, 6, 12]. Further, the temperature and