Optimal Control of a One-Dimensional Storage Process

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Abstract. We consider the discounted and ergodic optimal control problems related to a one-dimensional storage process. The existence and uniqueness of the corresponding Bellman equation and the regularity of the optimal value is established. Using the Bellman equation an optimal feedback control is constructed. Finally we show that under this optimal control the origin is reachable.

Introduction

We investigate the optimal control of a one-dimensional storage process. This problem arises in the economic planning of a nonrenewable natural resource (such as oil, mineral deposits or energy) in a socially managed economy. K. Arrow in [1] modelled the level of natural resource as a controlled jump-process. The randomness of the process was due to the uncertainty in the exploration of the natural resource. S. D. Deshmukh and S. R. Pliska studied a similar model in [5] under the assumption that the unexplored area has an infinite area. Let us briefly explain this model.

Let \( y(t) \) be the current level of the natural process at time \( t \geq 0 \). At each time \( t \) the planner determines the consumption rate \( c(t) \in [0, c_0] \), under which the storage level decreases with the rate \( c(t) \). Since the resource level is always non-negative, \( c(t) = 0 \) is the only choice whenever \( y(t) = 0 \). In addition to the consumption rate, the planner also determines the exploration rate \( e(t) \in [0, e_0] \) which is the intensity of search effort to discover additional sources of the

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resource. Under this policy the resource level has the jump rate \( \lambda(e(t)) \) and the jump-size distribution \( G(e(t), \cdot) \). Note that \( G(e, \cdot) \) has support on \([0, \infty)\).

In this paper we use the above model with feedback strategies. Let \( \pi(x) = (e(x), c(x)) \) be a Borel measurable map of \([0, \infty) \) into \([0, e_0] \times [0, c_0] \). The map \( \pi \) is an admissible strategy if (i) there is a unique storage process \( y(t) \) with the consumption rate \( c(y(t)) \) and the exploration rate \( e(y(t)) \) and the exploration rate \( e(y(t)) \) and (ii) \( c(0) = 0 \). For each admissible strategy \( \pi \) consider a discounted cost \( J^\alpha(x, \pi) \) with discount factor \( \alpha > 0 \).

\[
J^\alpha(x, \pi) = E \left[ \int_0^\infty e^{-\alpha t} (u(c(y(t)) - h(e(y(t)))) \, dt \mid y(0) = x \right]
\]

The optimal value \( v^\alpha(x) \) is the supremum of \( J^\alpha(x, \pi) \) over all admissible strategies. This problem is studied by S. R. Pliska [8] and S. D. Deshmukh and S. R. Pliska [5] in the case of no-holding cost \( f \). Heuristically \( v^\alpha \) satisfies the Bellman equation

\[
\sup_{0 \leq e \leq e_0, 0 < c \leq c_0} [(A^\alpha v^\alpha)(x) + u(c) - h(e)] = f(x) + \alpha v^\alpha(x)
\]

(0.2)

where \( A^\alpha \) is the infinitesimal generator of the storage process. Under (1.10)–(1.15) it is shown that the optimal value is bounded, continuous with bounded continuous derivative on \([0, \infty) \), this class of functions is denoted by \( C^1_0([0, \infty)) \). Moreover \( v^\alpha \) solves the integro-differential equation:

\[
\frac{d}{dx} v^\alpha(x) = \sup_{0 \leq e \leq e_0, 0 < c \leq c_0} \left\{ \frac{1}{c} \left[ u(c) - f(x) - \alpha v^\alpha(x) - h(e) + \int_0^\infty \left( v^\alpha(x + y) - v^\alpha(x) \right) \lambda(e) G(e, dy) \right] \right\}; \quad x > 0
\]

(0.3)

with boundary condition

\[
\alpha v^\alpha(0) = \sup_{0 \leq e \leq e_0} \left\{ -h(e) + \int_0^\infty \left( v^\alpha(y) - v^\alpha(0) \right) \lambda(e) G(e, dy) \right\}
\]

(0.4)

Note that (0.3)–(0.4) is in fact equivalent to (0.2). The unusual form of the boundary condition is caused by the state-space constraint.

By standard selection theorems one can choose \( \pi^* = (e^*, c^*) \) so that for all \( x \geq 0 \) \( e^*(x), c^*(x) \) maximizes (0.3)–(0.4). The properties of \( v^\alpha \) yield that \( \pi^* \) is admissible. Moreover if the consumption utility rate \( u \) is twice continuously differentiable around the origin then under the optimal strategy \( \pi^* \), the origin is reachable.