A VARIATIONAL DIFFERENCE SCHEME
FOR THE ONE-DIMENSIONAL DIFFUSION EQUATION

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In this paper we construct a homogeneous variational difference scheme for the diffusion equation assuming its coefficients to be bounded and measurable; the order of convergence of the scheme is $O(h^2)$.

We consider the boundary value problem

$$\frac{d}{dx} \left( K(x) \frac{du}{dx} \right) - g(x) u = - \frac{dF(x)}{dx}, \quad 0 < x < X,$$

subject to the boundary conditions

$$u(0) = a, \quad u(X) = b. \quad (2)$$

In [1, 2] a difference scheme was constructed with an estimate for the rate of convergence in $C$ of $O(h^2)$, wherein it was assumed that $K(x)$ is twice piecewise-differentiable and that $g(x)$ and $F(x)$ are piecewise-differentiable functions. In this paper, under much weaker conditions, we construct a difference scheme with a convergence rate of $O(h^2)$ in the $W^1$ grid norm.

We assume that $0 \leq g(x) \leq g_0 < \infty$, $0 \leq K_0 \leq K(x) \leq K_1 < \infty$, that $K(x)$ and $g(x)$ are measurable, and that $F(x)$ is of bounded variation.

By a solution of problem (1), (2) we shall mean a function $u(x) \in W^1$ which takes on the boundary values and satisfies the integral identity

$$\int_0^X \left[ K(x) \frac{du}{dx} \frac{d\varphi}{dx} + gu\varphi + F \frac{d\varphi}{dx} \right] dx = 0$$

for arbitrary $\varphi \in W^1$. It is a known fact that under these assumptions such a solution exists. Introducing the new independent variable $t(r) = \int_0^r \frac{ds}{K(s)}$, we have

$$\int_0^T \left[ \frac{du}{dt} \frac{d\varphi}{dt} + K(t) g(t) u(t) \varphi(t) + F(t) \frac{d\varphi}{dt} \right] dt = 0. \quad (3)$$

We have retained the old notation for the functions $u(t), K(t), g(t), \varphi(t), F(t)$.

§ 1. Construction of the Difference Scheme. Let $0 = x_0 < x_1 < \ldots < x_n = X$ be the nodes of the grid $t_k = t(x_k)$. We rewrite Eq. (3) in the form

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[ \frac{du}{dt} \frac{d\varphi}{dt} + Kgu\varphi + F(t) \frac{d\varphi}{dt} \right] dt = 0. \quad (4)$$

Let \( \Phi^h \) be the space of continuous functions \( \varphi(t) \), linear on each of the intervals \([t_{k-1}, t_k]\). We denote the subspace of functions belonging to \( \Phi^h \), for which \( \varphi(0) = \varphi(T) = 0 \), by \( \Phi_0^h \). Substituting \( \varphi(t) \) from \( \Phi^h \) into Eq. (4), we obtain an equation which the exact solution of the problem (1), (2) satisfies, namely,

\[
\sum_{k=1}^{n} \frac{u(t_k) - u(t_{k-1})}{\Delta_{k-1}} \varphi_{k-1} \Delta_{k-1} \varphi_{k} + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(u(t)) \varphi_{k-1} \Delta_{k-1} \varphi_{k} \frac{F(t)}{\Delta_{k-1}} dt + \sum_{k=1}^{n} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \int_{t_{k-1}}^{t_k} F(t) dt = 0.
\]

(5)

Here \( \varphi_k = \varphi(t_k) \), \( \Delta_{k-1} = t_k - t_{k-1} \).

We define the function \( u^h(t) \) as a function belonging to \( \Phi^h \) and satisfying Eq. (5), or, equivalently,

\[
\sum_{k=1}^{n} \frac{u(t_k) - u(t_{k-1})}{\Delta_{k-1}} \varphi_{k-1} \Delta_{k-1} \varphi_{k} + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(u(t)) \varphi_{k-1} \varphi_{k} \frac{F(t)}{\Delta_{k-1}} dt + \sum_{k=1}^{n} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \int_{t_{k-1}}^{t_k} F(t) dt = 0,
\]

(6)

Equating to zero the coefficients of \( \varphi_k \), \( k = 1, 2, \ldots, n-1 \), we obtain a system of equations in the quantities \( u_k \). We reduce this system to the final form

\[
\frac{1}{h_k} \left[ \frac{u_{k-1} - u_k}{h_k} - a_k \varphi_{k-1} \varphi_k + \frac{1}{h_k} \frac{u_{k+1} - u_k}{h_k} \right] - d_{k-1} u_{k+1} + d_k u_k - f_k = 0,
\]

(7)

where \( h_k = x_k - x_{k-1}, \ h_k = 0.5 (h_k + h_{k+1}) \),

\[
a_k = \frac{1}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx,
\]

and the coefficients \( c_k, d_k, f_k \) are given by the recursion relations

\[
d_{k-1} = \frac{1}{h_k} \left[ \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx + \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx \right],
\]

\[
d_k = \frac{1}{h_k} \left[ \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx + \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx \right],
\]

\[
f_k = \frac{1}{h_k} \left[ \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx + \frac{a_k}{h_k} \int_{x_{k-1}}^{x_k} g \left( \frac{x}{K(x)} \right) K(x) \, dx \right].
\]

We take the set of relationships (7), along with the conditions \( u_0 = a \), \( u_n = b \) as the approximating difference scheme for the initial problem on the grid with the nodes \( x_k \), the \( u_k \) are approximations for the values \( u(t_k) = u(x_k) \).

§ 2. Error Estimate. We put \( u_k = u(t_k) - u_k, \Delta \varphi_{k-1} = \varphi_k - \varphi_{k-1} \). Substituting Eq. (6) from Eq. (5), we obtain the relation

\[
\sum_{k=1}^{n} \frac{\Delta_{k-1}}{\Delta_{k-1}} \frac{\Delta \varphi_{k-1}}{\Delta_{k-1}} \varphi_{k} + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(u(t)) \Delta \varphi_{k-1} \varphi_{k} \frac{F(t)}{\Delta_{k-1}} dt = 0.
\]