A VARIATIONAL DIFFERENCE SCHEME
FOR THE ONE-DIMENSIONAL DIFFUSION EQUATION

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In this paper we construct a homogeneous variational difference scheme for the diffusion equation assuming its coefficients to be bounded and measurable; the order of convergence of the scheme is \( O(h^2) \).

We consider the boundary value problem

\[
\frac{d}{dx} \left( K(x) \frac{du}{dx} \right) - g(x) u = -\frac{dF(x)}{dx}, \quad 0 < x < X,
\]

subject to the boundary conditions

\[
u(0) = a, \quad u(X) = b.
\]

In [1, 2] a difference scheme was constructed with an estimate for the rate of convergence in \( C \) of \( O(h^2) \), wherein it was assumed that \( K(x) \) is twice piecewise-differentiable and that \( g(x) \) and \( F(x) \) are piecewise-differentiable functions. In this paper, under much weaker conditions, we construct a difference scheme with a convergence rate of \( O(h^2) \) in the \( W_1^1 \) grid norm.

We assume that \( 0 \leq g(x) \leq g_1 < \infty \), \( 0 < K_0 \leq K(x) \leq K_1 < \infty \), that \( K(x) \) and \( g(x) \) are measurable, and that \( F(x) \) is of bounded variation.

By a solution of problem (1), (2) we shall mean a function \( u(x) \in W_1^1 \) which takes on the boundary values and satisfies the integral identity

\[
\int_0^X \left[ K(x) \frac{du}{dx} + gu \varphi + F \frac{d\varphi}{dx} \right] dx = 0
\]

for arbitrary \( \varphi \in W_1^1 \). It is a known fact that under these assumptions such a solution exists. Introducing the new independent variable \( t(r) = \int_0^r \frac{ds}{K(s)} \), we have

\[
\int_0^T \left[ \frac{du}{dt} \frac{d\varphi}{dt} + K(t) g(t) u(t) \varphi(t) + F(t) \frac{d\varphi}{dt} \right] dt = 0.
\]

We have retained the old notation for the functions \( u(t), K(t), g(t), \varphi(t), F(t) \).

§ 1. Construction of the Difference Scheme. Let \( 0 = x_0 < x_1 < \ldots < x_n = X \) be the nodes of the grid \( t_k = t(x_k) \). We rewrite Eq. (3) in the form

\[
\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[ \frac{du}{dt} \frac{d\varphi}{dt} + K g(u) \varphi + F(t) \frac{d\varphi}{dt} \right] dt = 0.
\]
Let \( \Phi^h \) be the space of continuous functions \( \varphi(t) \), linear on each of the intervals \([t_{k-1}, t_k]\). We denote the subspace of functions belonging to \( \Phi^h \), for which \( \varphi(0) = \varphi(T) = 0 \), by \( \Phi^h \). Obviously, \( \Phi^h \in W_2^1 \). Substituting \( \varphi(t) \) from \( \Phi^h \) into Eq. (4), we obtain an equation which the exact solution of the problem (1), (2) satisfies, namely,

\[
\sum_{k=1}^{n} \frac{u(t_k) - u(t_{k-1})}{\Delta_{k-1}} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \Delta_{k-1} + \\
+ \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(t) \left[ u_{k-2}\varphi_{k-1}(t) + \varphi_k\beta_{k}(t) \right] dt + \\
+ \sum_{k=1}^{n} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \int_{t_{k-1}}^{t_k} F(t) dt = 0. 
\]

(5)

Here \( \varphi_k = \varphi(t_k) \), \( \Delta_{k-1} = t_k - t_{k-1} \),

\[
\alpha_{k-1} = \frac{t_k - t}{\Delta_{k-1}}, \quad \beta_k = \frac{t - t_{k-1}}{\Delta_{k-1}}.
\]

We define the function \( u^h(t) \) as a function belonging to \( \Phi^h \) and satisfying Eq. (5), or, equivalently,

\[
\sum_{k=1}^{n} \frac{u_k - u_{k-1}}{\Delta_{k-1}} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \Delta_{k-1} + \\
+ \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(u_{k-1}\alpha_{k-1} + u_k\beta_k)(\varphi_k\alpha_{k-1} + \varphi_k\beta_k) dt + \\
+ \sum_{k=1}^{n} \frac{\varphi_k - \varphi_{k-1}}{\Delta_{k-1}} \int_{t_{k-1}}^{t_k} F(t) dt = 0, 
\]

(6)

Equating to zero the coefficients of \( \varphi_k \), \( k = 1, 2, \ldots, n-1 \), we obtain a system of equations in the quantities \( u_k \). We reduce this system to the final form

\[
\frac{1}{h_k} \left[ a_k \varphi_k - u_k - u_{k-1} - a_k \frac{h_k - u_{k-1}}{h_k} \right] - \left( 1 - \frac{h_k}{h_k} \right) d_{k+1}u_{k+1} + c_k u_k + \frac{1}{h_k} \left. d_{k-1}u_{k-1} \right| = -f_k,
\]

(7)

where \( h_k = x_k - x_{k-1}, \quad h = 0.5 (h_k + h_{k+1}) \),

\[
a_k = \frac{1}{\int_{x_{k-1}}^{x_k} \frac{dx}{K(x)}}
\]

and the coefficients \( c_k, d_k, f_k \) are given by the recursion relations

\[
d_{k+1} = \left( \frac{a_k}{h_k} \right)^{2} \int_{x_{k-1}}^{x_k} \frac{dx}{K(x)} K(x) dx,
\]

\[
\varphi_k = \frac{1}{h_k} \left[ a_k \left( \int_{x_{k-1}}^{x_k} \frac{dx}{K(x)} \right)^{2} \right] - \left( \frac{a_k}{h_k} \right)^{2} \int_{x_{k-1}}^{x_k} \frac{dx}{K(x)} K(x) dx - \left( \frac{a_k}{h_k} \right)^{2} \int_{x_{k-1}}^{x_k} \frac{dx}{K(x)} F(x) dx.
\]

We take the set of relationships (7), along with the conditions \( u_0 = a, \quad u_n = b \) as the approximating difference scheme for the initial problem on the grid with the nodes \( x_k \), the \( u_k \) are approximations for the values \( u(t_k) = u(x_k) \).

\section{Error Estimate.}

We put \( r_k = u(t_k) - u_k, \Delta \varphi_k = \varphi_k - \varphi_{k-1} \). Subtracting Eq. (6) from Eq. (5), we obtain the relation

\[
\sum_{k=1}^{n} \frac{\Delta \varphi_k - \Delta \varphi_{k-1}}{\Delta_{k-1}} \Delta_{k-1} + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Kg(u(t) - u_{k-2}\varphi_{k-1} - u_k\beta_k) \varphi(t) dt = 0.
\]

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