Technical Note

On Stochastic Bang Bang Control

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Abstract. We consider a constrained stochastic control problem (stochastic bang-bang control) studied recently by many authors Ruzicka [1], Zzonkin-Krylov [2], and Ikeda-Watanabe [3], among others. Our techniques are quite different from theirs, and we obtain in addition some results complementing theirs which would appear to be of independent interest as well.

1. Recently many authors [1,2,3] have studied the following one-dimensional stochastic control problem, which, while of little practical utility, would appear to have many interesting theoretical features. Let \( W(t) \) denote a standard one-dimensional Wiener process, \( W(0) = 0 \), with \( F_w(t) \) the increasing sigma-algebra, and consider the simple stochastic equation:

\[
dx(t) = u(t)\,dt + dW(t); \quad x(0) = x, \quad 0 < t.
\] (1.1)

where the control \( u(t) \) is required to be measurable \( F_w(t) \) and such that

\[
|u(t)| < 1 \quad \text{a.e. with pr. one}
\] (1.2)

The problem is to minimise:

\[
\int_0^1 E(|x(t)|^2) \,dt
\] (1.3)

The immediate guess for the optimal control (by analogy with the deterministic case, or otherwise) is (bang-bang):

\[
u_0(t) = (-1) \text{sign} x_0(t)
\] (1.4)

with corresponding stochastic equation:

\[
dx_0(t) = (-1) \text{sign} x_0(t)\,dt + dW(t)
\] (1.5)

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The first bottle-neck was whether (1.5) had a strong solution or not. A special proof was given by Ruzicka [1] to establish strong solution, and later Zwonkin [4] proved the existence of strong solutions more generally for any bounded measurable drift. Ruzicka [1] went on to prove the optimality as well (in fact for the case of noisy observation), and later Ikeda-Watanabe [3] gave another proof using comparison theorems.

Our interest in the problem was prompted by the fact that if one uses the backward Kolmogorov equation for the conditional expectation:

\[ E(x_0(t)/x_0(0) = x) = u(t,x). \]  

we obtain the partial differential equation:

\[ \frac{\partial u}{\partial t} = (\text{sign } x) \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \]  

\[ u(0,x) = x. \]

But this equation has the obvious solution:

\[ u(t,x) = x - t \quad x > 0, t > 0 \]
\[ u(t,x) = x + t \quad x < 0, t > 0 \]

and this is clearly not the conditional expectation (1.6). In fact, we shall show here that:

\[ u(t,x) = \sqrt{1/(2\pi t)} \int_0^\infty y(\exp -t/2 + |x| - y^2/2t - x^2/2t) \]
\[ (\exp xy/t - (\exp -xy/t))dy \]

which does not satisfy (1.7). However, in the process of obtaining this, we are also able to give an independent proof of the optimality of (1.4), in addition to some other results which would appear to be not without interest. For example, it is well-known that

\[ \tilde{W}(t) = \int_0^t \text{sign } W(s) dW(s) \]

is also a Wiener process with respect to \( F_w(t) \), and is uncorrelated with \( W(t) \). But the quick conjecture that they are independent is false, as we show here, in particular, by calculating the joint distribution (which turns out to be singular with respect to Lebesgue measure).

The basic tool we shall use is the Ito differential rule applied to convex functionals, or alternately, the Doob-Meyer decomposition for submartingales.

2. Ruzicka [1] and Ikeda-Watanabe [3] consider the one-dimensional stochastic control problem described by the stochastic equation:

\[ dx(t) = u(t) dt + dW(t) \]

where \( W(t) \) is a standard Wiener process in \( \mathbb{R}_1 \), with respect to the growing sigma-algebra \( F_w(t) \), \( W(0) = 0 \), and the control \( u(t) \) is required to be measurable.