Selfinjective algebras of polynomial growth

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Dedicated to Hiroyuki Tachikawa for his 60th birthday

Introduction

Throughout the paper $k$ denotes a fixed algebraically closed field. We use the term algebra to mean finite dimensional $k$-algebra and the term module to mean finite dimensional left module. Algebras, as is usual in representation theory, are assumed to be basic and connected.

Representation-finite selfinjective algebras were classified by Riedtmann [Rd1; Rd2; Rd3; BLR] (see also [HW; W]). The classification splits into two cases: the standard algebras, which admit simply connected Galois coverings, and the remaining non-standard ones. The standard representation-finite selfinjective algebras $A$ are uniquely determined (up to isomorphism) by their Auslander-Reiten quivers $\Gamma_A$, and these are of the form $Z\Delta_c/\Pi$, where $\Delta$ is a Dynkin graph $A_n, D_n, E_6, E_7$ or $E_8$, $Z\Delta_c$ is the translation quiver $Z\Delta$ completed by a configuration $C$ of projective vertices, and $\Pi$ is a non-trivial admissible automorphism group of $Z\Delta_c$. The non-standard representation-finite selfinjective algebras occur only in characteristic 2, and there are classified by quivers and relations [Rd3; W].

Here, we are concerned with the problem of a classification of representation-infinite selfinjective algebras of polynomial growth, that is, algebras with infinitely many isomorphism classes of indecomposable modules and for which there is a natural number $m$ such that the indecomposable modules occur, in each dimension $d \geq 2$, in a finite number of discrete and at most $d^m$ one-parameter families (cf. [S1]).

In [T] Tachikawa proved that the trivial extension $T(H) = H \ltimes D(H)$ of a hereditary algebra $H$ of Euclidean type by its minimal injective cogenerator $D(H) = \text{Hom}_k(H, k)$ is 2-parametric, and hence representation-infinite selfinjective of polynomial growth. Recently, all representation-infinite trivial extensions $T(A)$ of polynomial growth have been classified in [ANS; NS; N]. In particular, it was shown that all such trivial extensions are standard.

In the paper, with the help of Galois coverings, we classify all representation-infinite selfinjective standard algebras of polynomial growth. The coverings used here are constructed from Euclidean and Ringel algebras. In contrast with the

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Riedtmann's classification of representation-finite selfinjective algebras, a complete description of the Auslander-Reiten quivers is here a consequence of the classification. In fact, the family of algebras $A_t = \mathrm{tk}_* \mathrm{ek}^*$, defined by two generators $x$, $y$ and the relations $x^2 = y^2 = yx - txy = 0$, shows that, for an arbitrary algebraically closed field $k$, there are infinitely many pairwise nonisomorphic standard selfinjective algebras of polynomial growth having the same Auslander-Reiten quiver.

1. The main result and the related background

1.1. Let $A$ be an algebra and $k[x]$ be the polynomial algebra in one variable. Following Drozd [D2] $A$ is called tame if, for any dimension $d$, there is a finite number of $A$-$k[x]$-bimodules $Q_i$, $1 \leq i \leq n_d$, which are finitely generated and free as right $k[x]$-modules, and satisfy the following condition:

(a) All but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ are of the form $Q_i \otimes k[x]/(x-\lambda)$ for some $\lambda \in k$ and some $i$, $1 \leq i \leq n_d$.

Let $\mu_A(d)$ be the least number of bimodules $Q_i$ satisfying the above condition. Then $A$ is called of polynomial (resp. linear) growth [S1] if there is a natural number $m$ such that $\mu_A(d) \leq d^m$ (resp. $\mu_A(d) \leq md$) for all $d \geq 2$. Further, $A$ is domestic [R1] if there is a finite number of $A$-$k[x]$-bimodules $Q_i$, $1 \leq i \leq n$, which are finitely generated free right $k[x]$-modules and satisfy the following condition:

(b) For each dimension $d$, all but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ are of the form $Q_i \otimes V$ for some $i$ and some indecomposable $k[x]$-module $V$.

$A$ is $n$-parametric if the minimal number of such bimodules is $n$.

1.2. Let $R$ be a locally bounded category [BG]. We denote by $\text{mod} R$ the category of all finite dimensional covariant functors from $R$ to the category of $k$-vector spaces. If $R$ is bounded (the number of objects is finite), then mod $R$ is equivalent to the category mod $A$ of left modules over the algebra $A = \oplus R$ formed by the quadratic matrices $a = (a_{xy})_{x,y \in R}$ such that $a_{xy} \in R(x,y)$. Conversely, to each algebra $A$ we can attach the bounded category $R$ with $A \cong \oplus R$ whose objects are formed by a complete set $E$ of orthogonal primitive idempotents $e$ of $A$, $R(e,f) = fAe$ and the composition is induced by the multiplication in $A$. We denote by $\Gamma_R$ the Auslander-Reiten quiver of $R[\text{G1}]$. The category $R$ is called tame (resp. domestic, of linear growth, of polynomial growth) if so is the algebra $\oplus A$ associated with every finite full subcategory $\Lambda$ of $R$ (cf. [DS1]). For a group $G$ of $k$-linear automorphisms of $R$ acting freely on the objects of $R$, we denote by $R/G$ the quotient category [G2] whose objects are the $G$-orbits of the objects of $R$. Then there is a Galois covering functor $F: R \to R/G$ which assigns to each object $x$ its $G$-orbit $Gx$. A locally bounded category $R$ is called simply connected [AS2] if it is triangular (its quiver has no oriented cycles) and, for any presentation $R \cong kQ/I$ of $R$ as a bound quiver category, the fundamental group $\Pi_1(Q,I)$ of $(Q,I)$ [MP] is trivial. It is equivalent (see [S3]) to the fact that $R$ is triangular and each Galois covering of $R$ is trivial. A locally bounded category $A$ is called standard if it admits a Galois covering $R \to A$ with $R$ simply connected. For locally representation-finite categories this definition