Exact Boundary Controllability on $L_2(\Omega) \times H^{-1}(\Omega)$ of the Wave Equation with Dirichlet Boundary Control Acting on a Portion of the Boundary $\partial \Omega$, and Related Problems*

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Abstract. Consider the wave equation defined on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. The control action is exercised in the Dirichlet boundary conditions only on $\Gamma_1$ and is of class $L_2(0, T; L_2(\Gamma_1))$; instead, homogeneous boundary conditions of Dirichlet (or Neumann) type are imposed on the complementary part $\Gamma_0$. The main result of the paper is a theorem which, under general conditions on the triplet $\{\Omega, \Gamma_0, \Gamma_1\}$ with $\Gamma_0 \neq \emptyset$, guarantees exact controllability on the space $L_2(\Omega) \times H^{-1}(\Omega)$ of maximal regularity for $T > T_0 > 0$, which depends on the triplet. This theorem generalizes prior results by Lasiecka and the author [L-T.3] (obtained via uniform stabilization) and by Lions [L.5], [L.6] (obtained by a direct approach, different from the one followed here). The key technical issue is a lower bound on the $L_2(\Sigma_1)$-norm of the normal derivative of the solution to the corresponding homogeneous problem, which extends to a larger class of triplets $\{\Omega, \Gamma_0, \Gamma_1\}$ prior results by Lasiecka and the author [L-T.3] and by Ho [H.1].

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1. Introduction; Literature and Motivation; Statement of Main Results

1.1. Introduction

Throughout this paper \( \Omega \) is an open, bounded domain in \( \mathbb{R}^n \), with sufficiently smooth boundary \( \partial \Omega = \Gamma \). In \( \Omega \) we consider the following nonhomogeneous Dirichlet problem for the wave equation in the solution \( w(t, x) \):

\[
\begin{align*}
\left\{
\begin{array}{ll}
\frac{\partial^2 w}{\partial t^2} - \Delta w & \text{in } (0, T) \times \Omega = Q, \\
w(0, \cdot) = w^0, & \quad w_1(0, \cdot) = w^1 \quad \text{in } \Omega, \\
w|_{\Sigma_0} = 0 & \quad \text{in } (0, T) \times \Gamma_0 = \Sigma_0, \\
w|_{\Sigma_1} = u & \quad \text{in } (0, T) \times \Gamma_1 = \Sigma_1,
\end{array}
\right.
\end{align*}
\]

(1.1a)

(1.1b)

(1.1c)

(1.1d)

with control function \( u \in L_2(0, T; L_2(F_1)) \subseteq L_2(\Sigma_1) \). Here, \( \Gamma_0 \) and \( \Gamma_1 \) are subsets of \( \Gamma \), with \( \Gamma_0 \cup \Gamma_1 = \Gamma \), \( \Gamma_0 \) possibly empty \( \emptyset \), while \( \Gamma_1 \) is nonempty and relatively open. We also set \( (0, T) \times F = \Sigma \). In qualitative terms, the problem of exact boundary controllability for (1.1) is as follows. Given the triplet \( \{\Omega, \Sigma_0, \Sigma_1\} \) we ask whether there is some \( T_0 > 0 \) (depending on the geometry of the triplet) such that if \( T > T_0 \), then the following steering property of (1.1) holds true: for all initial data \( w^0, w^1 \) in some preassigned space \( Z = Z_1 \times Z_2 \), there exists a suitable control function \( u \in L_2(0, T; L_2(F_1)) \) whose corresponding solution of (1.1) satisfies \( w(T, \cdot) = w(T, \cdot) = 0 \). If this is the case, we say more precisely that the dynamics (1.1) is \textit{exactly controllable in the space }\( Z \text{ over the time interval } [0, T] \) (be means of \( L_2 \)-controls). In Section 5 we also consider the case where (1.1c) is replaced by \( \partial w/\partial v|_{\Sigma_0} = 0 \).

1.2. Literature and Motivation

There exists a large body of literature regarding exact controllability problems for the wave equation (or other hyperbolic dynamics) with control action either in the Dirichlet boundary control (B.C.) as in (1.1c-d), or else in the Neumann B.C. (or variations thereof). Among the numerous works on the nonhomogeneous \textit{Dirichlet} case prior to 1985, we cite only a few: Russell [R.2], [R.3], Littman [L.7], [L.8], Lagnese [L.1], [L.2], Ralston [R.1], and others, and we refer to the other works quoted in these papers. It is important to observe, however, that in all of the literature prior to 1985, it was always assumed that the space \( Z \) of arbitrary initial data \( w^0, w^1 \) be "smooth" in one way or another; more precisely, it was assumed that

\[
(w^0, w^1) \in Z = H^2(\Omega) \times H^1(\Omega)
\]

(1.2a)

or

\[
(w^0, w^1) \in Z = H^1(\Omega) \times L_2(\Omega)
\]

(1.2b)

under appropriate geometric restrictions on \( \Omega \) [R.3], [L.1], [L.8] ¹

1 Steering any initial condition (i.c.) in \( Z \) to rest \( (0, 0) \) in \( [0, T] \) is equivalent—by time reversibility of (1.1)—to steering the origin \( (0, 0) \) to any (target) final condition (f.c.) in \( Z \) in \( [0, T] \); hence, to steering any i.c. in \( Z \) to any f.c. in \( Z \) over \( [0, T] \), by \( L_2(\Sigma_1) \)-controls.