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Symplectic topology as the geometry of generating functions

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Introduction

In the last years important progress in symplectic topology have been achieved using either of the following approaches:

The Gromov-Floer approach studies pseudo holomorphic spheres in symplectic manifolds to infer results on the symplectic topology of the manifold. In particular, results on the geometry of Lagrange submanifolds, existence of fixed points of symplectomorphisms isotopic to the identity, have thus been obtained for which we refer to [G1, Fl1, Fl2]. This approach is essentially the only one available when working in arbitrary symplectic manifolds. In a certain sense we might say that this approach considers *symplectic topology as the study of pseudo-holomorphic curves in almost complex manifolds.*

The other approach, which is the one we are interested in here, could be called the Conley-Zehnder approach. It is mainly concerned with periodic orbits of Hamiltonian systems. This can be a very efficient way of dealing with symplectic topology questions, at least in $\mathbb{R}^{2n}$, as has been sufficiently demonstrated for instance by Ekeland and Hofer in [E-H2, E-H3, V3] and the author [V2]. The method is based on studying the topology of the action functional, this is what we now explain. Let $H(t,x)$ be a time dependent Hamiltonian on $\mathbb{R}^{2n}$, we search 1-periodic solutions of $\dot{x}=J\nabla H(t,x)=X_H(t,x)$ where $dH(t,x)\xi=\omega(X_H(t,x),\xi)=\langle J\nabla H(t,x),\xi\rangle$ (if we identify $\mathbb{R}^{2n}$ to $\mathbb{C}^n$, then J is the matrix of multiplication by i). Such periodic solutions can be obtained as critical points of the action functional:

$$A_H(x)=\int_{S^1}\langle Jx,\dot{x}\rangle-H(t,x)dt\,.$$

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The usual way to get these critical points is by taking minmax on some classes of sets invariant by the flow. While this would be ackward to describe in this introduction, we point out that it becomes the usual minmax theory if we replace A_H by a finite dimensional reduction A_H^N. The capacities of H are then the critical value associated to a minmax on certain cohomology classes (cf. [V3]). It is thus fundamental to compute the topology of the sublevel sets $\{x \mid A_H^N(x) \leqq \lambda\}$. Thus we can say that this approach considers *symplectic topology as the topology of the action functional.*

Our approach is based on the remark that the action functional is nothing else than a special generating function (for the definition of this we refer to Sect. 1). This was already exploited in [V1] to show that all the indices defined for periodic solutions of Hamiltonian systems are, up to a constant, all equal, and that this indices have a very geometric definition. The natural generalization of such a result is to show that the sublevel sets of a generating function are, after a suitable suspension, diffeomorphic (provided we restrict ourselves to generating functions with quadratic phase cf. Sect. 1). Thus we can get rid of action functionals, and work with any generating function of the time one flow of the Hamiltonian, which is a much more flexible tool. The ideas developed here originated from a question that Yasha Eliashberg asked to Helmut Hofer and myself. Hofer's answer is contained in [H]. The present paper is an expanded version of our own answer which is Corollary 4.8. In the course of the proof which was based on the usual study of the "action functional topology" it appeared that chosing between the "broken geodesic" reduction of Chaperon-Laudenbach-Sikorav, and the Lyapounov-Schmidt reduction of Conley-Zehnder, meant having to chose between two equally desirable properties, C^0 continuity of the capacities (Proposition 4.6), and the "triangle inequality" (Proposition 4.8). The advantage of our approach is that we get both properties, moreover most of the proofs are then obvious. The paper is organized as follows.

In Sect. 1 we define generating functions, and prove that they are essentially unique.

Section 2 associates to every Lagrange submanifold L of T^*B and cohomology class u in $H^*(B)$ a real number $c(u, L)$. In this section and the next one we prove basic properties of this numbers.

In Sect. 4 we associate to every compact supported symplectomorphism ψ of $\mathbb{R}^{2n}$ isotopic to the identity (in the set of compact supported symplectomorphisms) a Lagrange submanifold Γ_ψ of T^*S^{2n} which is the compactification of its graph. The construction of Sect. 3 then yields two numbers that we shall denote by $c_+(\psi)$ and $c_-(\psi)$.

This allows us to define positive symplectomorphisms [by the condition $c_-(\psi)=0$] and to prove several remarkable properties of these maps.

We also prove that a compact supported symplectomorphism has infinitely many periodic points in the interior of its support.

Section 5 is devoted to studying the behaviour of our invariants by symplectic reduction. The treatment is in no way complete and is only pursued as needed to give a simple proof of the "camel problem", Sect. 6 deals with a property of "simple hypersurfaces".