Geometric Conditions for Optimization in a Linear Space

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Communicated by A. V. Balakrishnan

ABSTRACT

In this paper we give new properties of convex functions whose domains are subsets of a linear space; we use them in order to get geometric characterizations of a minimant of a convex function.

1. Introduction. In his book [8], Laurent gives some conditions in order to affirm when a point is a minimant of a convex function whose domain is a topological linear space; he adopts a geometric point of view by working with the notion of “cône de déplacement”. In this paper we intend to improve this idea. First we are working in an arbitrary real linear space (without topology); after Klee [6] and Köthe [7] we introduce for an arbitrary set the operators of affine hull, intrinsic core and algebraic hull which are very geometric and useful. We consider also real functions whose domains do not necessarily coincide with the whole space; these functions are convex or strictly quasi-convex. Finally we explain our results in terms of cones which are very simple and intuitive as the “visual cone” introduced by Bragard [4].

2. Generalities. 2.1. Throughout this note we shall work in a real nonzero linear space $E$ and we shall adopt notations of previous papers [1, 2, 3]. More particularly, we shall consider for a subset $A$ of $E$ the following sets: $\mathcal{A}$—the affine hull of $A$, i.e. the set of all points of the form $\sum_{i=1}^{n} \alpha_i x_i$ with $x_i \in A$ and $\sum_{i=1}^{n} \alpha_i = 1$; $\mathcal{B}$—the intrinsic core of $A$, i.e. the set $\{x \in E : \forall y \in A, \exists \varepsilon(x,y) > 0, x + r(y - x) \in A, \forall r \in [-\varepsilon, +\varepsilon]\}$; $\mathcal{C}$—the algebraic hull of $A$, i.e. the set $A \cup \{x \in E : \exists y \in A, y \neq x, [y : x] \subseteq A\}$. A set $A$ is said to be properly convex if $A$ is convex and if $A = E$; $A$ is algebraically open (resp. properly open) if $\mathcal{A} = A$ (resp. $\mathcal{C} = A$ and $A = E$). It is clear that a convex set $A$ is properly open if and only if $A \cap \mathcal{C} = \emptyset$ for each set $B$ which does not meet $A$.

2.2. Bragard has introduced the visual cone $V(A, x)$ of a set $A$ since a point $x$ if $x \in A$ (resp. $x \notin A$), then $V(A, x)$ is the union of all the half-lines $[x : u]$ (resp. $]x : u[$) which contain a segment $[x : u]$ (resp. $]x : u[)$ included in $A$ [4]. It is not
difficult to prove that a point \( x \) of a set \( A \) lies in the intrinsic core of \( A \) if and only if \( V(A,x) = A \).

In some problems it is more interesting to use a similar cone: if \( x \in bA \), then \( P(A,x) \) is the union of \( \{x\} \) and of all half-lines \([x:u]\) which contain a nontrivial segment \([x:u]\) included in \( bA \).

Elementary geometric reasonings give relations between the two cones \( V(A,x) \) and \( P(A,x) \): "If \( A \) is a convex algebraically open set and \( x \in bA \), then \( V(A,x) = P(A,x) \). In corollary if \( A \neq \emptyset \) and \( x \in bA \), then \( V(A,x) = P(A,x) \) and \( V(A,x) \) is algebraically open. If \( A \) is star-shaped with respect to the point \( x \) (briefly, star-shaped at \( x \)), then \( P(A,x) \subseteq V(A,x) \). If \( x \) lies in a convex set \( A \) whose intrinsic core is nonempty, then \( bP(A,x) \subseteq bV(A,x) \). Moreover, one has always \( bV(A,x) \subseteq bP(A,x) \).

The proofs of these statements are very simple and are left to the reader.

2.3. A finite family \( \mathcal{C} = \{A_0,A_1,\cdots,A_n\} \) is said to be properly separated if there exist linear functionals \( f_0,f_1,\cdots,f_n \) and real numbers \( \lambda_0,\lambda_1,\cdots,\lambda_n \) such that

a) \( \sum_{i=0}^{n} \lambda_i = 0 \); b) \( \sum_{i=0}^{n} \lambda_i < 0 \); c) \( A_j \subseteq \{ x \in E : f_j(x) < \lambda_j \} \) for all \( j \in \{0,1,\cdots,n\} \); d) there exists \( j_0 \in \{0,1,2,\cdots,n\} \) such that \( f_{j_0} \neq 0 \) and \( A_{j_0} \subseteq \{ x : f_{j_0}(x) = \lambda_{j_0} \} \) [2].

If we adopt a reasoning found in Halkin's paper [5], we find a supplementary statement:

Let \( A_0 \) be a nonempty convex set, \( A_1,A_2,\cdots,A_n \) convex sets whose intrinsic core is nonempty and whose affine hull coincides with the whole space \( E \). The family \( \mathcal{C} = \{A_0,A_1,\cdots,A_n\} \) is properly separated if and only if \( A_0 \) does not meet \( \cap_{j=1}^{n} A_j \).

**Proof.** First we show that the condition is necessary. Suppose there exists a point \( x_0 \) in \( A_0 \cap (\cap_{j=1}^{n} A_j) \). Since the family \( \mathcal{C} \) is properly separated, there exist linear functionals \( f_j \) and real numbers \( \lambda_j \) such that \( A_j \subseteq \{ x \in E : f_j(x) < \lambda_j \} \) (\( j = 0,1,\cdots,n \)). But it is possible to find a number \( p \) of \( \{1,2,\cdots,n\} \) such that \( f_p \neq 0 \): in these conditions, \( A_p \subseteq \{ x \in E : f_p(x) < \lambda_p \} \) because \( A_p \) is properly convex, whence \( f_p(x_0) < \lambda_p \) and \( \sum_{j=0}^{n} f_j(x_0) < \sum_{j=0}^{n} \lambda_j \leq 0 \); this contradicts the equality \( \sum_{j=0}^{n} f_j(x_0) = 0 \).

Conversely consider the sets \( M = A_1 \times A_2 \times \cdots \times A_n \) and \( N = A_0 \cap \{(x_1,x_2,\cdots,x_n) \in E^n : x_1 = x_2 = \cdots = x_n \} \) in the space \( E^n \): \( N \) is a nonempty convex set and \( M \) is a convex set whose affine hull coincides with \( E^n \) and whose intrinsic core is nonempty. The sets \( M \) and \( N \) can be properly separated [7; p. 187] in the space \( E^n \) by a hyperplane and the family \( \mathcal{C} \) is thus properly separated [2; p. 12]. \( \square \)

2.4. Throughout this paper, \( f \) will be a real function whose domain is a subset of the linear space \( E \); the set where \( f \) is defined will be denoted by \( \text{dom} f \) and called the domain of \( f \). If \( \alpha \) is a real number, then \( F(\alpha) \) will be the set \( \{ x \in \text{dom} f : f(x) < \alpha \} \). If the eventual minimum of \( f \) on a subset \( A \subseteq \text{dom} f \) is reached by a point \( a \) of \( A \), then the point \( a \) will be called a minimant of \( f \) on \( A \).

A function \( f \) will be convex if \( \text{dom} f \) is a nonempty convex subset of \( E \) and if \( f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \) for all points \( x,y \) of \( \text{dom} f \) and each \( \lambda \in [0,1] \). Remark that a convex function \( f \) in our sense gives a proper convex function \( \tilde{f} \) considered by Rockafellar [10] (it suffices to take \( \tilde{f}(x) = f(x) \) if \( x \in \text{dom} f \) and \( f(x) = +\infty \) if \( x \not\in \text{dom} f \)) and conversely: these two notions are thus equivalent.