On Likelihood Robustness

Yu. A. Rozanov

International Institute for 
Applied Systems Analysis 
2361 Laxenburg, Austria

Communicated by A. V. Balakrishnan

Let

\[ \xi(t) = a(t) + \eta(t), \quad t \in T, \]

be a standard "signal plus noise" model where \( a(t) \) and \( \eta(t) \) are independent Gaussian processes on an abstract set \( T \) (the process \( a(t) \) can be deterministic).

ally an observer is quite uncertain about a probability distribution \( \nu \) of the noise \( \eta(t) \), \( t \in T \). With respect to this point A. V. Balakrishnan* has recently posed a question about asymptotic behavior of the likelihood "signal/noise" ratio \( \frac{d\nu}{d\nu^0} \), when \( \nu \) approaches a proper "white noise" distribution \( \nu^0 \) from below:

\[ \pi = \log \frac{d\nu}{d\nu^0} = \frac{1}{2} \log \left( \frac{d\nu}{d\nu^0} \right) = \frac{1}{2} \log \left( \frac{d\nu}{d\nu^0} \right) \]

or for any \( \nu = \{A_k\} \); here \( R, R_0 \) are the corresponding correlation matrices of zero mean variables \( \eta(t) \). For example in a case of the generalized processes on a Hilbert space the correlation operators \( R \) might approach the identity.

One can see a difficulty to formulate an exact question because of the fact that different distributions \( \nu \) are orthogonal so there is not an obvious base to consider \( \frac{d\nu}{d\nu^0} \) convergence as functions of "elementary events". Nevertheless there is a phenomenon which one may consider as a robustness property of the likelihood ratio \( \frac{d\nu}{d\nu^0} \). Let us set

\[ \pi = \log \frac{d\nu}{d\nu^0} - E \log \frac{d\nu}{d\nu^0} \]

and let \( \pi_0 \) be determined in a similar way with respect to the distributions \( \nu^0 \), \( \nu^0 \). There is a natural way of imbedding \( \pi_0 \) into the Hilbert \( L^2 \)-space with a norm

\[ \| \cdot \|_p = \int |\cdot|^2 d\nu. \]

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*Personal communication.
For the variables \( \pi \) as functionals of the observing process \( \xi(t), t \in T \), we have

\[
\lim \| \pi - \pi_0 \|_P = 0. \quad (3)
\]

For details we refer to the book [1] and start with a case when the "signal" \( a(t) \) is deterministic. As known in this case the variable \( \pi \) is a linear functional of the process \( \xi(t), t \in T \), in a sense that \( \pi \) belongs to the linear closure \( H^1_\xi \) of all linear forms

\[
A_\xi = \sum_k A_k \xi(t_k).
\]

Moreover there is a one-to-one bounded mapping

\[
A_\xi \leftrightarrow A\eta \quad (4)
\]

which helps to identify the corresponding \( \pi \) as

\[
\pi \leftrightarrow \frac{h}{\|h\|_P^2},
\]

where \( h \) is the minimal element of the hyperplane

\[
l(\cdot) = 1;
\]

this linear continuous functional on \( H^1_\eta \) is determined by the signal \( a(t), t \in T \), as

\[
l[\eta(t)] = a(t), \quad t \in T.
\]

We ought to note that under the condition (1) the mapping (4) is one-to-one bounded uniformly over all \( R, R \uparrow R_0 \), because

\[
\|A\xi\|^2_P = \|A\eta\|^2_P + l(A\eta)^2.
\]

Let \( \| \cdot \|_{P^0} \) be the \( L^2 \)-norm with respect to \( P^0 \). Because of condition (1) we have

\[
\|A\eta\|_P \leq \|A\eta\|_{P^0}
\]

so any \( P^0 \)-fundamental sequence of linear forms, \( A_n\eta, n = 1, 2, \ldots \), is \( P \)-fundamental which gives us a natural way of imbedding \( \pi_0 = \lim_{n \to \infty} A_n \xi \) into the space \( H^1_\xi \) mentioned above; we have

\[
\pi_0 \leftrightarrow \frac{h_0}{\|h_0\|_{P^0}^2}
\]