A Variational Inequality Approach to Constrained Control Problems for Parabolic Equations

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Abstract. A distributed optimal control problem for parabolic systems with constraints in state is considered. The problem is transformed to control problem without constraints but for systems governed by parabolic variational inequalities. The new formulation presented enables the efficient use of a standard gradient method for numerically solving the problem in question. Comparison with a standard penalty method as well as numerical examples are given.

1. Introduction

Starting with the works of Yvon [19] and Mignot [7] much interest is given for the optimal control problems governed by variational inequalities, both from a theoretical and a numerical point of view. See the recent book of Barbu [1] for the elliptic and parabolic case, as well as [8]–[17] for parabolic and hyperbolic problems.

Under certain assumptions, a classical remark shows that variational inequalities are equivalent to a minimization problem with constraints. Similarly, we prove that there is a close connection between control problems governed by variational inequalities and constrained control problems (constraint in state). We even have, in special cases, equivalence between the two types of problems.

This gives a new interpretation of optimal control problems governed by variational inequalities and provides a new approximation of constrained control problems.
In order to make the above ideas clear we study the following model problem $(P)$:

Let $V, H, \mathcal{U}$ be Hilbert spaces with dense and compact imbedding $V \subset H \subset V^*$, and $A: V \to V^*, B: \mathcal{U} \to H$ be linear, continuous operators such that

\begin{align}
(Au, u) \geq \omega |u|^2_V, & \quad \omega > 0, \quad u \in V, \\
(Au, v) = (u, Av), & \quad y, v \in V,
\end{align}

where $(\cdot, \cdot)$ is the pairing between $V$ and $V^*$ (if $v_1, v_2 \in H$ then $(v_1, v_2)$ is the inner product in $H$) and $|\cdot|^2_V$ is the norm in the Banach space $V$.

Consider the control problem

$(P)$ Minimize \( \{J(y, u) = g(y) + h(u)\}$

subject to

\begin{align}
y' + Ay &= Bu + f \quad \text{a.e. in } [0, T], \\
y(0) &= y_0, \\
y(t) &\in C \quad \text{in } [0, T].
\end{align}

Here $C \subset H$ is a closed, convex subset, $y_0 \in C, Ay_0 \in H, f \in L^2(0, T; H)$, $g: L^2(0, T; H) \to \mathbb{R}$ is convex, continuous, majorized from below by a constant $c$, and $h: L^2(0, T; \mathcal{U}) \to ]-\infty, +\infty]$ is convex, lower semicontinuous, proper, satisfying

\[ \lim_{|u|_{L^2(0, T; \mathcal{U})} \to \infty} h(u) = +\infty. \]

Under the above hypotheses, for any fixed $u \in L^2(0, T; \mathcal{U})$, equation $(1.3)$, $(1.4)$ has a unique solution $y \in C(0, T; V), y' \in L^2(0, T; H)$ and $(1.5)$ makes sense. As usual, we have denoted by $C(0, T; H)$ the space of all continuous functions from $[0, T]$ to $H; L^p(0, T; V)$ is the space of all (classes of) Lebesgue measurable functions $[0, T] \to V$ such that

\[ \int_0^T \|y(t)\|_V^p \, dt < \infty \]

with usual modification if $p = \infty$.

If we also have control constraints $u \in \mathcal{U}_0$ (a closed, convex subset of $L^2(0, T; \mathcal{U})$), this may be implicitly expressed by adding to $h$ the indicator function of $\mathcal{U}_0$.

We assume the existence of an admissible pair $[\tilde{y}, \tilde{u}]$ for $(P)$. This assumption may be relaxed, according to Section 3. It is easy to show the existence of at least one optimal pair $[y^*, u^*]$.

The plan of the paper is as follows. Section 2 contains the main result. In Section 3 we discuss two special cases. Section 4 is devoted to the analysis of an algorithm for solving problem $(P)$. Finally, in the last section we give numerical examples by which we demonstrate the advantage, for example, of the proposed method over the standard penalty technique.

Several results of this paper were announced in [17]. Finally, we remark that the methods presented here can be applied to optimal shape design problems; especially to the important family of design problems with constraints in state. This will be discussed in a forthcoming paper.