On Differentiability with Respect to Parameter of Solutions to Convex Optimal Control Problems Subject to State Space Constraints

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Abstract. A family of convex optimal control problems that depend on a real parameter \( h \) is considered. The optimal control problems are subject to state space constraints.

It is shown that under some regularity conditions on data the solutions of these problems as well as the associated Lagrange multipliers are directionally-differentiable functions of the parameter.

The respective right-derivatives are given as the solution and respective Lagrange multipliers for an auxiliary quadratic optimal control problem subject to linear state space constraints.

If a condition of strict complementarity type holds, then directional derivatives become continuous ones.

1. Introduction

Dependence on parameters of solutions to constrained optimal control problems have been studied mostly in the sense of differentiability properties of so called extremal value function, which to every parameter assigns the corresponding optimal value of the cost functional (cf. eg. [4, 9]).

On the other hand some papers ([1, 2]) deal with such global properties of solutions to these problems as continuity or Lipschitz continuity with respect to parameters.

Only a few papers are devoted to differential properties of solutions to constrained optimal control problems, or similar problems of optimization in functional spaces. One has to mention here, the results due to A. Haraux [7] and F. Mignot [10] concerning directional differentiability with respect to parameters.
of solutions to variational inequalities. The authors exploited the idea of directional differentiability of projections on convex sets in Hilbert spaces.

J. Sokołowski [11] extended the result of [7, 10] to the case where the bilinear form in variational inequality depends on a parameter. He used this result to obtain sensitivity results for distributed parameter quadratic control problems subject to linear control constraints.

In [8] directional differentiability of parametric convex optimal control problems, subject to control constraints is proved. In the proof the structure of the systems plays an important role. Namely the optimality conditions in the form of maximum principle are treated as some parametric convex programming problems. Using the form of directional derivatives for such problems it is shown that the right-derivatives of solutions to constrained optimal control problems exist and are given by the solutions to some quadratic optimal control problems. Unfortunately this approach can not be applied directly to convex optimal control problems with state space constraints. The difficulties arise in getting convergence of dual variables in appropriate topology. To overcome these difficulties we take advantage of conditions of optimality in the form of stationarity of a Lagrangean associated with the problem of optimal control.

In contradistinction to majority of the papers we consider the state space constraints in the Sobolev space $W^{1,1}(0, T)$ instead of $C(0, T)$. Accordingly the Lagrangean assumes a little bit different form, more suitable for our purpose.

Using the results of regularity of solutions due to W. W. Hager [6] and their Lipschitz continuity with respect to a parameter [2] we are able to prove directional differentiability with respect to the parameter of optimal primal and dual variables for convex optimal control problems with state space constraints.

The right-differentials of primal and dual variables are given as optimal primal and dual variables for some quadratic optimal control problems with linear state space constraints.

To simplify the exposition problems with linear state space constraints are considered although the obtained results can be extended to convex constraints, satisfying some regularity conditions.

Note that due to technical difficulties we are not able to derive similar results for problems with state and control constraints, although using a formal argument it is easy to obtain the tentative form of the right-differential for such problems.

Some notations used:

- $R^n$ is $n$-dimensional Euclidean space with the inner-product denoted by $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$
- $L^p(0, T), 1 \leq p < \infty$ are Banach spaces of functions integrable on $(0, T)$ with power $p$, supplied with the usual norm denoted by $\|\cdot\|_p$
- $L^\infty(0, T)$ is a Banach space of functions essentially bounded on $(0, T)$ with the norm $\|f\|_\infty = \operatorname{esssup}_{t \in [0, T]} \|f(t)\|$

If $f \in L^p(0, T), g \in L^q(0, T)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, then $\langle f, g \rangle = \int_0^T \langle f(t), g(t) \rangle \, dt$