On those Principles of Mechanics which depend upon Processes of Variation.*)

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Chief among the questions of mathematical interest, and the points of controversy, about the variational principles of mechanics are those concerning the exact meaning of the process of 'variation' used; — whether it can be regarded as a literal (proper) variation, as considered in the calculus of variations, or not. These questions are:

(a) As to the relation of the principle of least action to Hamilton's principle, and the connected one as to whether the variable t is to be varied in the former (in the latter, δt = 0);

(b) As to the extent of the principles, — whether, and under what conditions, they are applicable to the cases of non-existence of a force-function, non-holonomous conditions, and conditions which depend on t explicitly, — and the connected difficulty as to the transformation of the principles from rectangular to general coordinates.**

Since the time of Lagrange, both questions have arisen, in a more or less explicit form, in the work of Rodrigues, Jacobi, Ostrogradsky, Routh, A. Mayer, Sloudsky, Bertrand, Voss, Helmholtz, Réthy, Hölder, Appell, and myself. The historical part, I propose to deal with elsewhere; here I will try, with especial reference to some recent papers of Réthy***), to give what seem to me to be satisfactory answers to both questions.


**) I find it convenient to call general coordinates 'generalised' ones when they are mutually independent.

With regard to the questions (a), various views as to the relation of the principle of least action to Hamilton's principle have been taken by different people, or by the same person at different times. Owing to the impossibility of preserving the mutual independence of the variations of the generalised coordinates with the supposition that the quantity $T - U$ is to be unvaried in the variation when $t$ is taken as the independent variable (so that $\delta t = 0$), Jacobi gave the principle of least action a form in which $t$ is eliminated, so that it is quite distinct from, and less general then, Hamilton's principle*). Secondly, in view of this, Lagrange's principle of least action seemed to Ostrogradsky and to Mayer in his earlier (1877) work**) to be an incorrect formulation of Hamilton's principle. Thirdly, Rodrigues, in a memoir of 1814 which had been overlooked by most until much later, had shown that the above difficulty could be surmounted, and Lagrange's principle conceived as one quite different from Hamilton's, but approaching it — and deviating from Jacobi's — in form, just as Lagrange seems to have intended, if only we vary $t$ (do not assume $\delta t = 0$). This view was accepted by Routh and Sloudsky, and by Mayer in 1886, with explicit abandonment of his former view. Fourthly, there is Helmholtz's view that Hamilton's principle is a form of Lagrange's principle. Helmholtz is only right if suggestion of Hamilton's principle contained in the equations of condition do not depend explicitly on $t$. And lastly, there is a perfectly correct identity, established by Réthy, and showing more clearly this result of Helmholtz's.

If the equations of condition do not contain $t$ explicitly, we have, as Réthy remarked,

$$\delta T = \sum \frac{\partial T}{\partial q_v} \delta q_v + \sum \frac{\partial T}{\partial \dot{q}_v} \frac{\delta q_v}{\delta t} - 2T \frac{\delta t}{\delta t},$$

where the $q_v$'s are generalised coordinates. Putting

$$\delta_t T = \delta T + 2T \frac{\delta t}{\delta t},$$

*) In this form, 'the principle of least action' has been given in most text-books since Jacobi's time; for example, Darboux, 'Leçons sur la théorie générale des surfaces', t. 2, Paris 1889, pp. 491—500; Appell, 'Traité de mécanique rationnelle', 2e ed., t. 2., 1904, pp. 425—429; and Maggi, 'Principii della teoria matematica del movimento dei corpi', Milano, 1896, pp. 394—396.