Semisimplicial Objects and the Eilenberg-Zilber Theorem

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The object of this paper is to do an explicit computation which arises in connection with reduced powers in homological algebra, and in particular in determining $P^0$. (The computation shows that $P^0 = 1$ in the topological case.) We are also able to solve a problem of Dold [1] p. 273. Dold asks when an FD-complex with a commutative associative diagonal has $P^0 = 1$, and produces an example "from nature" where $P^0 \neq 1$. This paper also gives a way of constructing Steenrod operations, when one has a pair of adjoint functors and suitable axioms are satisfied. This is done in Chapter 8 of the preceding paper. Dold [1] has shown how to use the methods described here to construct Steenrod operations in the conventional topological situation.

We have borrowed shamelessly from Dold [1] and Dold and Puppe [2]. The second paper cited gives Carter's version of the Eilenberg-Zilber Theorem, which we reproduce here. We also give yet another (yet really the same) proof of the famous normalisation theorem of Eilenberg and MacLane [5]. In justification we must point out that the theorems which we repeat here, do not appear in the literature in sufficiently functor-theoretical language for our purposes. We were reluctant to impose the burden of translation on readers and have taken it on ourselves.

§ 1. Category Theory

Let $\mathcal{C}$ be an arbitrary category. We construct a category $\mathcal{C}^+$ as follows. The objects of $\mathcal{C}^+$ are the same as those of $\mathcal{C}$ and $\text{Hom}_{\mathcal{C}^+}(A, B)$ is the free abelian group on $\text{Hom}_{\mathcal{C}}(A, B)$. Composition of morphisms is defined so that it is bilinear and so that it agrees with the composition in $\mathcal{C}$ on the generators of the free abelian groups.

Let $\mathcal{D}$ be a category such that for any two objects $A, B \in \mathcal{D}$, $\text{Hom}(A, B)$ is an abelian group and composition is bilinear. We say $\mathcal{D}$ is pre-0-additive. For any category $\mathcal{C}$, $\mathcal{C}^+$ is pre-0-additive. We form a new category $\mathcal{D}^0$ by adjoining a zero object 0. We have to add to the category exactly two morphisms for each object $A$ of $\mathcal{D}$, namely the members of the one element sets $\text{Hom}(0, A)$ and $\text{Hom}(A, 0)$. We also have the one-element set $\text{Hom}(0, 0)$. We now form an additive category $\mathcal{D}^0$ (that is, we adjoin finite direct sums) as follows. The objects of $\mathcal{D}^0$ are $k$-tuples of objects of $\mathcal{D}^0$, which we think of as arranged in a column.
with \( k \) rows \((k = 1, 2, 3, \ldots)\). A morphism from a \( k \)-tuple to an \( s \)-tuple is an \((s \times k)\)-matrix whose entries are morphisms in \( \mathbb{D}^0 \) (with appropriate domain and range objects). We shall write \( \mathbb{C}^\oplus \) instead of \( \mathbb{C}^{+\oplus} \) and \( \mathbb{C}^0 \) instead of \( \mathbb{C}^{+\!0} \) if there is no possibility of confusion.

1.1. Any functor from \( \mathbb{C} \) to an additive category \( \mathbb{A} \) has a unique additive extension from \( \mathbb{C}^+ \) to \( \mathbb{A} \). Any additive functor from \( \mathbb{D} \) to \( \mathbb{A} \) has an additive extension to \( \mathbb{D}^\oplus \) which is unique (up to natural isomorphism).

Remark. Any pre-0-additive category \( \mathbb{D} \) can be embedded as a full subcategory of an abelian category. See FREYD [3] pp. 112–115.

1.2. Let \( \mathbb{D} \) and \( \mathbb{B} \) be pre-0-additive categories. We define \( \mathbb{D} \otimes \mathbb{B} \) to be the category whose objects are pairs \((D, P)\) with \( D \in \mathbb{D} \) and \( P \in \mathbb{B} \) and whose morphisms from \((D, P)\) to \((D', P')\) are \( \text{Hom}(D, D') \otimes \text{Hom}(P, P') \). We write the object \((D, P)\) as \( D \downarrow \mathbb{B} \).

2. The Semisimplicial Category

Let \([n]\) be the set of integers \( \{0, 1, \ldots, n\} \). The semisimplicial category \( \mathbb{S} \) has as its objects the sets \([n]\) for \( n = 0, 1, \ldots \) and as morphisms the (weakly) monotone functions \([m] \rightarrow [n]\). We have the usual monotone functions

\[
\begin{align*}
\varepsilon_i^\circ &: [n-1] \rightarrow [n] \\
\eta_i^\circ &: [n-1] \rightarrow [n]
\end{align*}
\]

leaving \( i \in [n] \) uncovered and covering \( i \in [n] \) twice.

We shall prove the normalisation theorem, concerning the structure of \( \mathbb{S}^{-}\!0 \), where \( \mathbb{S}^{-}\! \) is the dual of \( \mathbb{S} \). In \( \mathbb{S}^{-}\!0 \) we have the objects \([n]\) for \( n \geq 0 \) and the zero object \( 0 \). If \( n, m \geq 0 \) then the group of morphisms from \([m]\) to \([n]\) is the free abelian group on the monotone functions \([n] \rightarrow [m]\). It will be convenient to put \([n]^{-\!}=0\) if \( n<0 \). If \( n>0 \) and \( 0 \leq i \leq n \), we define

\[
\partial_i^\circ: [n]^{-\!}\rightarrow [n-1]^{-\!}
\]

corresponding to \( \varepsilon_i^\circ \). If \( 0 \leq i \leq n \), we define

\[
s_i^\circ: [n]^{-\!}\rightarrow [n+1]^{-\!}
\]

corresponding to \( \eta_i^\circ \). We now define \( \partial_i^\circ: [n]^{-\!}\rightarrow [n-1]^{-\!} \) and \( s_i^\circ: [n]^{-\!}\rightarrow [n+1]^{-\!} \) to be zero for all other integers \( i \) and \( n \).

2.2. Definition. A semisimplicial object in a category \( \mathbb{C} \) is a contravariant functor \( \mathbb{S} \rightarrow \mathbb{C} \). If \( \mathbb{C} \) is pre-additive, this is the same as an additive covariant functor \( \mathbb{S}^\!^{-\!0} \rightarrow \mathbb{C} \). A dual semisimplicial object is obtained by interchanging “covariant” and “contravariant” above.