Comment on Schöner-Haken systematic adiabatic approximation for stochastic differential equations

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Recently, G. Schöner and H. Haken (SH) have proposed a nice systematic elimination procedure and a nice systematic adiabatic approximation (SAA) for stochastic differential equations (SDE's). In this comment, some simplified operator calculi and some general formulas for SH's theory are obtained for an important class of SDE's.

1. Introduction

In [1, 2], Schöner and Haken have dealt with systems governed by nonlinear SDE's of the type

\[ dq = h_1(q, t) \, dt + df(q, t) \]  

where \( q \) is the state vector of the system, \( h_1 \) is a function that contains, in general, nonlinearities of \( q \), and \( df \) is the Stratonovich (or Ito) differential of a stochastic force

\[ df = H_2(q, t) \, dw. \]  

Here \( w \) is a Wiener process and the diffusion matrix \( H_2 \) may, in general, be state dependent.

A large class of physical and non-physical systems that show spontaneous formation of spatial or temporal structure can be described by stochastic dynamics of the type (1.1) [3–6]. The stochasticity may result from microscopic degrees of freedom within the system (internal noise) or from coupling to a fluctuating environment (external noise). In both cases, state dependent diffusion coefficients may arise [7].

In SH's method, instability is defined with respect to a basic deterministic solution \( q_0 \) of (1.1). Fluctuations are treated as perturbations of that solution. They expand (1.1) around that basis solution and diagonalize with respect to the linear part of the resulting equations for the deviations from \( q_0 \). So they can distinguish the linearly damped (or slaved) variables \( s \) and the linearly undamped variables (or order parameters) \( u \). It is at the non-equilibrium phase transition that the first linearly unstable modes occur. The form of the resulting equations will in general be [3]:

\[ du = A_u u \, dt + q_0(u, s, t) \, dt + F_u(u, s, t) \, dw \]  
\[ = \tilde{q}_0(u, s, t) \, dt + F_u(u, s, t) \, dw \]  

\[ ds = A_s s \, dt + p_0(u, s, t) \, dt + F_s(u, s, t) \, dw. \]  

Where \( u \in \mathbb{R}^d \) are the order parameters and \( s \in \mathbb{R}^n \) are the slaved variables. The Jordan matrices \( A_u \) and \( A_s \) have eigenvalues \( \lambda_u \) and \( \lambda_s \) respectively with \( \text{Re} \lambda_u \geq 0; \text{Re} \lambda_s < 0 \). The functions \( q_0 \) and \( p_0 \) contain nonlinear, deterministic terms, while \( F_u \in \mathbb{R}^d \times \mathbb{R}^m \) and \( F_s \in \mathbb{R}^n \times \mathbb{R}^m \) are the coefficients of the stochastic differentials \( dw \) of \( m \) independent Wiener processes \( w \in \mathbb{R}^m \).

The slaving principle provided a systematic approach towards elimination of all the stable modes of a system consisting of a set of nonlinear SDE's. And hence the asymptotic behavior of the system is determined by an equation which contains the unstable modes \( u \) only. Schöner and Haken have presented a systematic and constructive procedure to eliminate the slaved variables starting from the basic assumption [1, 2]

\[ s = s(t, \text{chance}) = s(u(t), t, z^{(v)}(v = 2, 3, \ldots)). \]  

As a result of (1.5) the functions \( q_0, p_0, F_u \) and \( F_s \) are also functions of \( u(t) \) and \( z^{(v)}(v = 2, 3, \ldots) \) only.

In this comment, we given some simplified operator calculi and some general formulas for SH's theory for a class of important SDE's.

2. Simplified operator calculi to SH's new adiabatic approximation for a class of SDE's

For general (nonlinear) case, SH have obtained successfully the results [1, 2]

\[ s = \sum_{m=2}^{\infty} \sum_{m} = c^{(m)} \]
\[ c^{(m)} = G_0 \left( p_0^{(m)} d \tau + F_s^{(m)} d w_t \right) - G_0 \sum_{n=1}^{m-2} d^{(n)} c^{(m-n)} \]

and defined a new adiabatic approximation as follows:

\[ s_{ad}^{(m)} = \sum_{m=2}^{\infty} c_{ad}^{(m)} \]

\[ c_{ad}^{(m)} = G_0 \left( p_0^{(m)} d \tau + F_s^{(m)} d w_t \right) \]

For simplicity, we only consider the scalar case below. To obtain the \( s_{ad} \), we find that when the equation of \( \sigma_t \) is linear in \( \sigma_t \), we need not evaluate \( c_{ad}^{(m)} \) and then sum it but can use, instead, the operator equation to obtain \( s_{ad} \) directly from \[ s_{ad} = G_0 \left( p_0 d \tau + \sum_{i=1}^{m} F_i^t d w_t^i \right). \] (2.1)

Where, according to the SH theory,

\[ G_0 \ldots d \tau = \int_{-\infty}^{t} e^{\lambda(t-\tau)} d \tau, \]

\[ G_0 \ldots d w_t^i = \int_{-\infty}^{t} e^{2\zeta(t-\tau)} d w_t^i \]

and \( u_t \) in (2.1) in the SAA can be viewed as a parameter that is time and chance independent.

We assume

\[ p_0 = A(u_t) s_{ad} + a(u_t); \quad F_i^t = B_i(u_t) s_{ad} + b_i(u_t) \]

(2.3)

\( w_t^i \) are independent Wiener processes.

Using that, in (2.1) according to the SAA, \( G_0 \) does not operate on \( u_t \) we obtain from (2.1) and (2.3) the following operator equation

\[ s_{ad} = \left[ I - A(u_t) G_0 d \tau - \sum_{i=1}^{m} B_i(u_t) G_0 d w_t^i \right]^{-1} \]

\[ a(u_t) G_0 d \tau + \sum_{i=1}^{m} b_i(u_t) G_0 d w_t^i \]

(2.4)

To show the usefulness of (2.4), we calculate some model systems below.

1. The Haken-Zwanzig model with additive noise

\[ d u_t = (\alpha u_t - a u_t s_t) d t + F_u d w_t^1 \]

(2.5)

\[ d s_t = (-\beta s_t + b u_t^2) d t + F_s d w_t^{2(1)} \]

(2.6)

In this equations \( \alpha > 0, \beta > 0, a, b \) are real constants and \( a \) and \( b \) have the same sign. \( w_t^{(1)} \) and \( w_t^{(2)} \) are two independent Wiener processes. Obviously \( u_t \) is the order parameter and \( s_t \) the slaved process. We assume \( \alpha/\beta \leq 1 \).

Take SAA, by virtue of (2.4), we have from (2.6)

\[ s_{ad} = \frac{b u_t^2}{\beta} G_0 d \tau + F_s G_0 d w_t^{2(2)} \]

or

\[ s_{ad} = \frac{b u_t^2}{\beta} G_0 d \tau + F_s G_0 d w_t^{2(2)} \]

(2.7)

where

\[ z_t^{(2)} = \int_{-\infty}^{t} e^{-\beta(t-\tau)} d w_t^{(2)}. \] (2.8)

Substituting (2.7) into (2.5) we can get the order parameter equation in the SAA.

2. The Haken-Zwanzig model with multiplicative noise

\[ d u_t = (\alpha u_t - a u_t s_t) d t + F_u d w_t^{1(1)} \]

(2.9)

\[ d s_t = (-\beta s_t + b u_t^2) d t + F_s s_t d w_t^{2(2)} \]

(2.10)

This model is the same as (2.5), (2.6) except that the term \( F_s s_t d w_t^{2(2)} \) in (2.6) becomes the term \( F_s s_t d w_t^{2(2)} \) in (2.10). Under the SAA, we have from (2.4)

\[ s_{ad} = [I - F_s G_0 d w_t^{2(2)}]^{-1} G_0 u_t^2 d \tau \]

(2.11)

\[ z_t^{(m)} = \int_{-\infty}^{t} e^{-\beta(t-\tau)} z_t^{(m-1)} d w_t^{(2)} \quad (m = 3, 4, \ldots) \] (2.12)

with \( z_t^{(2)} = 1 \).

Equation (2.11) agrees with (6.16) of [1]. From (2.11) and (2.9) the order parameter equation obtained.

3. The single-mode laser model: additive noise

The following set of equations has the structure of the single-mode laser equations [2]

\[ d u_t = (\alpha u_t + a u_t s_t) d t + F_u d w_t^{1(1)} \]

(2.13)

\[ d s_t = (-\beta s_t - b u_t^2) d t + F_s s_t d w_t^{2(2)} \]

(2.14)

Here \( \alpha > 0, \beta > 0, a, b, c, F_u \) and \( F_s \) are constants and \( w_t^{(1)} \), \( w_t^{(2)} \) are two independent Wiener processes. The order parameter \( u_t \) corresponds to the mode amplitude of the light field and \( s_t \) to the inversion. For simplicity, we do not consider the vectorial feature of the light field. We assume \( \alpha/\beta \leq 1 \).

Take the SAA, we obtain from (2.4)

\[ s_{ad} = \frac{b u_t^2}{\beta} G_0 d \tau + F_s G_0 d w_t^{2(2)} \]

(2.15)

By expanding the second term of (2.15), we get

\[ s_{ad} = \frac{b u_t^2}{\beta} G_0 d \tau + F_s \sum_{k=1}^{\infty} (-b u_t^2)^{k-1} z_t^{(2k)} \] (2.16)