Necessary Conditions for Multiple Integral Problem in the Calculus of Variations

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1. Introduction

Let \( \Omega \) be a bounded and open subset of the Euclidean space \( \mathbb{R}^n \) with a sufficiently smooth boundary. We shall study here the following multiple integral problem in the calculus of variations:

\[
\min \left\{ \int_\Omega L(y(x), y'(x)) \, dx ; y \in K \right\}
\]  

(1.1)

where \( K \) is either all of \( W^{1,p}_0(\Omega) \) or the closed convex subset \( \{ y \in W^{1,p}_0(\Omega) ; y(x) \geq \varphi(x) \text{ a.e. } x \in \Omega \} \) (the "obstacle" problem).

There exists an extensive literature on necessary conditions of optimality for problem (1.1) in the case where \( K = W^{1,p}_0(\Omega) \) and \( L \) is either differentiable (see e.g. [12, 15]) or a convex integrand [7]. Recently, Clarke [5] has considered the general case where the integrand \( L \) is locally Lipschitz and derived an analogue of the classical Euler-Lagrange conditions of optimality expressed in term of the generalized gradient of \( L \). Here, by a different method already used by the author in other contexts (see [1, 2]) one extends the Clarke optimality theorem in [5] to general problems of the form (1.1) (see Theorem 1 below). The same approach is used to obtain in Theorem 2 a similar result for integrands \( L \) of the form (3.5).

The plan of the paper is the following.

The main results of this paper, Theorems 1 and 2, are stated in Sect. 3 and proved in Sects. 4 and 5 respectively. These theorems are used to derive in Sect. 6 some new existence for nonlinear elliptic variational inequalities.

2. Preliminaries

Throughout this paper \( \Omega \) will be a bounded and open subset of the Euclidean space \( \mathbb{R}^n \) with the boundary \( \Gamma \) sufficiently smooth. By \( L^p(\Omega ; \mathbb{R}^m) ; 1 \leq p \leq \infty \), we shall denote the space of all \( p \)-summable vectorial functions \( y : \Omega \to \mathbb{R}^m \). By \( C(\bar{\Omega} ; \mathbb{R}^m) \) we shall denote the space of all continuous functions \( y : \bar{\Omega} \to \mathbb{R}^m \). For
\( m = 1 \) we shall simply write \( L^p(\Omega) \) and \( C(\Omega) \). For any natural number \( k \) and \( 1 \leq p \leq \infty \) we shall denote by \( W^{k,p}(\Omega) \), \( W_0^{k,p}(\Omega) \), and \( W^{-k,p}(\Omega) \) the usual Sobolev spaces on \( \Omega \). We shall write \( H^k(\Omega) = W^{k,2}(\Omega) \) and \( H_0^k(\Omega) = W_0^{k,2}(\Omega) \).

For a given real valued function \( y \) on \( \Omega \) we shall denote by \( V_y(\mathbf{x}) \) the gradient of \( y \) at \( \mathbf{x} \).

Let \( \varphi : \mathbb{R}^m \to \mathbb{R} \) be a locally Lipschitzian function. By the Rademacher theorem \( \varphi \) is a.e. differentiable on \( \mathbb{R}^m \) and obviously the function \( V\varphi \) is measurable and essentially bounded on every bounded subset of \( \mathbb{R}^m \). We associate with \( \varphi \) the multivalued mapping \( D\varphi \) defined by

\[
D\varphi(y) = \bigcap_{\delta > 0} \bigcap_{v(N) = 0} \text{conv} \{ \nabla \varphi(S(y, \delta) \cap N) \}
\]

where \( S(y, \delta) \) is the ball of radius \( \delta \) and center \( y \) in \( \mathbb{R}^m \) and \( v \) is the Lebesgue measure in \( \mathbb{R}^m \).

Such a mapping [see also formula (6.2) below] arises in theory of generalized solutions for ordinary differential equations (see [9]).

The generalized gradient of \( \varphi \) at \( y \), denoted \( \partial \varphi(y) \) is the set (see [5, 14])

\[
\partial \varphi(y) = \text{conv} \{ w \in \mathbb{R}^m; w = \lim_{\mathbf{y}_n \to y} V\varphi(y_n) \}
\]

(2.2)

It turns out that \( D\varphi(y) = \partial \varphi(y) \) for all \( y \in \mathbb{R}^m \). We also recall that if \( \varphi \) is a convex function then \( \partial \varphi \) is just the subdifferential of \( \varphi \), i.e.,

\[
\partial \varphi(y) = \{ w \in \mathbb{R}^m; \varphi(y) \leq \varphi(z) + \langle w, y - z \rangle \}; \quad \forall z \in \mathbb{R}^m \}.
\]

(2.3)

In the sequel we shall denote by the same symbol \( | \cdot | \) the norm in \( \mathbb{R}^n \), \( \mathbb{R}^{m+1} \), and \( \mathbb{R}^m \) and by \( \langle \cdot, \cdot \rangle \) the usual inner product in these spaces.

3. The Main Results

Consider the minimization problem (1.1) where \( K \) is the closed convex subset of \( W_0^{1,p}(\Omega) \), \( 1 \leq p \leq \infty \)

\[
K = \{ y \in W_0^{1,p}(\Omega); y(x) \geq \psi(x) \text{ a.e. } x \in \Omega \}.
\]

Here \( \psi \in C(\Omega) \) is a given function satisfying the condition

\[
\psi(x) \leq 0 \text{ for all } x \in \Gamma.
\]

The particular case \( K = W_0^{1,p}(\Omega) \) which corresponds to \( \psi \equiv -\infty \) will be allowed.

As regards the integrand \( L : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) the following condition will be imposed

(i) \( L(y, z) \geq 0 \) for all \( (y, z) \in \mathbb{R} \times \mathbb{R}^n \), and for some constant \( M \), for all \( (y, z) \) and \( (u, v) \) in \( \mathbb{R} \times \mathbb{R}^n \), we have

\[
L(y + u, z + v) \leq \exp(M(|u, v|)) \left[ L(y, z) + M(|u, v|)(1 + |(y, z)|) \right].
\]

(3.1)

A condition of this type has been already used by Clarke [5] and we must notice that it implies that the function \( L \) is locally Lipschitzian and

\[
|\nabla L(y, z)| \leq M(L(y, z) + |(y, z)| + 1), \text{ a.e. } (y, z) \in \mathbb{R} \times \mathbb{R}^n.
\]

(3.2)