The algebra of many-valued quantities.

Von

Rosalind Cecily Young in Cambridge (England).

The following theory is one that I have recently adopted for the better treatment of theories involving limits[1]), where it has grown increasingly inconvenient to have to consider separately the upper or lower or other individual values of a numerical limit which is not unique[2]), for want

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[1]) A preliminary treatment was embodied in my Dissertation for the Ph. D., Cambridge, "Foundations for the generalisation of the Theory of Stieltjes Integration etc An n-dimensional treatment" (1929) and indicates the main features of the theory. Refinements introduced into the present exposition may be summarised as follows.

1. In the concept of *many-valuedness*, a symbol a (now a *quantity*), instead of being identified with a *set* (of values), is now conceived as having *any one* of a given set of values *collectively considered*,—in contradistinction to the idea of a variable, which assumes *individually considered values* in a given range which is generally fixed.—The exact nature of the concept, as of any mathematical entity, is best understood from the uses to which the concept is put, and in this case these are quite different from manipulations of sets.

2. By the introduction of the quantity having *no* values, o, the new *nought* (without prejudice to the "zero" (0) of our ordinary numberscheme), several simplifications are rendered possible; and *inter alia*

3. the definition of a *link* of two quantities (having the values common to both) as precisely complementary to that of their *union* (which has all the values of either); and the purer conception of the process of *levelling* (suppressing all values numerically $> K$).

4. The explicit definition of an *infinitesimal* also simplifies the exposition.

5. I absolutely exclude any reference to "infinite values", pending the precise definition and theory of such values, which will form the subject of a later paper. In accordance with this, the treatment of limits is that of *finite limits* throughout, i. e. concerns exclusively the finite values of limits, which may *or may not* constitute the *complete limits*. On this point, the present treatment is a good deal more precise than the original one.

[2]) The general idea of considering *all the limits*, and not merely upper and lower limits, seems to have been first utilized by W. H. Young in 1908: "Sulle due funzioni a più valori costituite dai limiti d' una funzione di variabile reale a destra ed a sinistra di ciascun punto", Rend. Acc. Lincei (5) **17**, 582—87.

of accurate rules for their collective manipulation[3]). Applications of the theory will be published elsewhere; the theory, however, seems to be of sufficient interest in itself. As an illustration of the efficiency of the new instrument, the rules given by Theorem V (p. 283), and more generally by Theorems VI and VII, should be compared, also for elegance and precision, with the current inequalities (which they of course include):

$$\underline{\lim}_{m\to\infty} a_m + \underline{\lim}_{m\to\infty} b_m \leqq \underline{\lim}_{m\to\infty} (a_m + b_m) \leqq \underline{\lim}_{m\to\infty} a_m + \overline{\lim}_{m\to\infty} b_m \leqq \overline{\lim}_{m\to\infty} (a_m + b_m)$$
$$\leqq \overline{\lim}_{m\to\infty} a_m + \overline{\lim}_{m\to\infty} b_m,$$

and the corresponding ones for products.

1. Many-valuedness.

When a symbol $a, b, x, f(x)$, etc. represents *any one* of a given *set of numbers,* we say that it represents, or *is*, a (*finite*) *quantity,* and the given numbers are called its *values.*

In the particular case when there is only one number given, the quantity is said to be *one-valued* and is identified with the given number.

The necessary pendent of the notion of a quantity with more than one value is that of a quantity *without any values*; this then has to be classed with many-valued quantities in the same way as the null-set has to be classed with sets generally. We shall call it *nought*[4]) and denote it by

$$\bar{o}.$$

A quantity with at least one value is therefore said to be *non-nought.*

A quantity with a bounded set of values only is said to be *strictly finite.* A quantity with positive values only is said to be *positive,* and if its values have a positive lower bound it is more specifically described as *strictly positive.* Similarly, a quantity with negative values only is said to be *negative,* and if its values have a negative upper bound, it is said to be *strictly negative.*

A quantity none of whose values is 0, is said to be *definite.* If it does not have values as near as we please to 0, it is said to be *strictly* definite.

If any of these properties belongs to a given one-valued quantity, it belongs to it of course strictly.

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[3]) Such relations as appear in the paper of W. H. Young just quoted, e. g.

$$F_L(P) < n, \quad H_R(P) \leqq n, \quad G_R(P) + k < F_L(P) < H_R(P) - k,$$

for his many-valued (right- and left-hand limiting) functions, tacitly assume some rudiments of an algebra of many-valued quantities, though the relations in this case being of so simple a nature, there was no inconvenience or ambiguity in introducing them.

[4]) The ordinary 0 is called *zero.*