Rings of Invariants and Linkage of Determinantal Ideals

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Introduction

In the first part of this paper we give a criterion for when a ring of invariants under a finite Abelian group action is a polynomial ring (Theorem 1.2). The criterion involves conditions on the depth of certain conormal modules and the first Koszul homology, and is mainly interesting as a negative condition: if a ring of invariants happens to be singular, then Theorem 1.2 yields strong obstructions for deforming and linking this singularity [12, 3, 4, 15]. As a first application, we compute the non-Cohen-Macaulay loci for conormal modules of symmetric determinantal ideals (Theorem 1.7).

In the second part we use the previous results to show some “rigid” behavior of (symmetric) determinantal ideals under linkage (excluding the trivial cases where the ideal is a complete intersection, or has grade 2). For example, if two determinant \( P \)-ideals \( I \) and \( J \), one of which is generic, belong to the same linkage class, then \( P/I \) and \( P/J \) have to be isomorphic (Theorem 2.8) (but \( I \) and \( J \) are not necessarily equal, cf. Example 2.16). We also show the following: let \( I \) be a symmetric determinantal ideal, let \( J \) be a determinantal ideal, and assume that \( I \) or \( J \) is generic. Then \( I \) and \( J \) are in different linkage classes (Theorem 2.10). Somewhat weaker results are obtained in the non-generic case (Theorem 2.2). Although we restrict ourselves to determinantal and symmetric determinantal ideals, one can apply the same methods to obtain similar results for ideals defined by Pfaffians (the non-Cohen-Macaulay loci of their conormal modules can be read from [1]).

As an application, we construct infinitely many arithmetically Gorenstein curves in \( \mathbb{P}^5 \) (arithmetically Cohen-Macaulay curves in \( \mathbb{P}^4 \)) which are projectively equivalent but belong to different linkage classes (Theorem 2.6), contrasting the well known fact that there exists only one linkage class of arithmetically Gorenstein curves in \( \mathbb{P}^4 \) (arithmetically Cohen-Macaulay curves in \( \mathbb{P}^3 \)) [34, 8] (C. Huneke independently showed that there are infinitely many linkage classes of arithmetically Cohen-Macaulay curves in \( \mathbb{P}^4 \)).

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As another application we give a criterion for when certain ideals of grade 3 in the linkage class of a generic (symmetric) determinantal ideal are again (symmetric) determinantal (Corollary 2.14). Contrasting a result of Bertini and Del Pezzo on classifying projective varieties of minimal degree (cf. Theorem 2.12), we indicate a simple construction for obtaining reduced graded Cohen-Macaulay algebras of minimal multiplicity whose defining ideals are neither determinantal nor symmetric determinantal (Corollary 2.14c, Example 2.15).

The paper concludes with a question on the smoothability of determinantal ideals (Question 3.1), which is answered in the affirmative for the case of maximal minors (Proposition 3.3).

Throughout the paper we will use the following notations and conventions: Let $I$ be an ideal in a power series ring $P$ over a field $k$, or a homogeneous ideal in a polynomial ring $P$ over $k$, set $R = P/I$, and let $M$ be a finitely generated $R$-module. We denote by $\dim R$ the Krull dimension, by $\text{edim}(R)$ the embedding dimension, by $e(R)$ the multiplicity, by $\text{Quot}(R)$ the total ring of quotients, and by $\omega_R$ the canonical module of $R$. We write $v(M)$ for the minimal number of generators of $M$, and we say that $M$ has a rank and that $\text{rank}_R M = n$, if $M \otimes_R \text{Quot}(R) \cong (\text{Quot}(R))^n$.

In the given situation, the concepts of grade and height $(ht I)$ of an ideal $I$ coincide, and $I$ is called perfect if $R$ is Cohen-Macaulay. $I$ (or $R$) is a complete intersection if $v(I) = \text{grade } I$, and $I$ (or $R$) is generically a complete intersection if $I$ is unmixed and $v(I_q) = \text{grade } I_q$ for all $q \in \text{Ass } R$. We define the following closed subsets of $\text{Spec } P$: $V(I) = \{q | q \supseteq I\}$, $\text{NCI}(I) = \{q | I_q \text{ not a complete intersection}\}$, $\text{Sing}(R) = \{q | R_q \text{ not regular}\}$, and $\text{NCM}(M) = \{q | M_q \text{ not Cohen-Macaulay}\}$. For the codimension of a closed set $V$ in $\text{Spec } R$ we write $\text{codim}_R(V)$.

The module $\frac{I/I^2}{(I/I^2 \otimes \omega_R)}$ is called the (twisted) conormal module of $I$. By $H_i(I; P)$ we will denote the $i$th Koszul homology with respect to a (minimal) system of generators of $I$. Serre's conditions are written as $(R_i)$ and $(S_i)$ [21]. A finite extension $R \subset S$ of normal domains is called divisorially unramified, if it is unramified in codimension one. A linear automorphism $g \in \text{GL}(n, k)$ is said to be a pseudo-reflection if $g$ has finite order and $\text{rank}_{k}(g - \text{id}) \leq 1$.

A projective curve in $\mathbb{P}^n_k$ will be $\text{Proj } R$, where $R$ is a graded factor ring of $k[X_0, \ldots, X_n]$, $R$ is generically a complete intersection, and $\dim R = 2$. Two projective curves $\text{Proj } R$ and $\text{Proj } R'$ in $\mathbb{P}^n_k$ are projectively equivalent if there exists an isomorphism $R \cong R'$ induced by a linear automorphism on $k[X_0, \ldots, X_n]$. The degree of $\text{Proj } R$ is defined as $e(R)$.

Two $P$-ideals $I$ and $J$ are said to be linked if there exists a regular sequence $q \subseteq I \cap J$ with $J = (q): I$, and $I = (q): J$. We say that a $P$-ideal $J$ is in the linkage class of $I$ and write $J \in L(I)$, if there exists a series of $P$-ideals $K_0, \ldots, K_n$ with $K_0 = I$, and $K_n = J$, such that $K_{i+1}$ and $K_i$ are linked ($0 \leq i < n$). For basic properties of linkage we refer to [22].

Let $A$ be a $r$ by $s$ matrix with entries in $P$ ($r \leq s$). Then $I_t(A)$ denotes the $P$-ideal generated by all $t$ by $t$ minors of $A$. We set $I_t(A) = P$, if $t < 1$, and $I_t(A) = 0$ if $t > r$. We say that $I$ is a determinantal $P$-ideal (or symmetric determinantal $P$-ideal respectively) if there exist integers $1 \leq t \leq r \leq s$ and a (symmetric) $r$ by $s$ matrix $A$ with entries in the maximal ideal of $P$ or homogeneous entries in the irrelevant