The $\bar{\partial}$-Neumann Operator on the Unit Ball in $\mathbb{C}^n$

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1. Introduction

The $\bar{\partial}$-Neumann problem was introduced in the fifties as a means for proving existence theorems for holomorphic functions on complex manifolds (Kohn and Spencer [3]). Since then, the main applications to complex analysis have centered around the solution of the Cauchy-Riemann equations $\bar{\partial}u = f$ which arises from the solution of the $\bar{\partial}$-Neumann problem. This problem was first solved on strictly pseudoconvex domains by Kohn [4], who proved existence and regularity properties of the $\bar{\partial}$-Neumann operator $N$ (the definition and formal properties of $N$ are briefly reviewed in Sect. 2). The operator $\bar{\partial}^*N$ then solves the $\bar{\partial}$-equation

$$\bar{\partial}(\bar{\partial}^*Nf) = f,$$ (1.1)

provided $f$ is $\bar{\partial}$-closed and orthogonal to the space of harmonic forms.

In this paper we first show how the above program can be reversed: given the operator $S = \bar{\partial}^*N$, one can obtain $N$ by the simple formula

$$N = SS^* + S*S.$$ (1.2)

Even though (1.2) is very natural and easy to prove, it apparently has not appeared in the literature before.

As an application we present an explicit integral formula for $S = \bar{\partial}^*N$ on the ball; via (1.2) one therefore obtains a simple description for the Neumann operator as well. Our formula is based on our recent joint work with Lieb [6], where we had obtained an integral operator $T_q : L^2_{0,q+1}(D) \to L^2_{0,q}(D)$ on a strictly pseudoconvex domain which agrees with $\bar{\partial}^*N_{q+1}$ on range $\bar{\partial}$ in case $D$ is a ball. Here we show that, in this case, $T_q = \bar{\partial}^*N_{q+1}$ on all of $L^2_{0,q+1}$. It appears likely that the method presented here can be combined with the results in [6] to show that $T_q$ describes the principal part of $\bar{\partial}^*N$ in the general case as well, at least for $D \subseteq \mathbb{C}^2$. We hope to investigate this question at some other time.

Recently there has been much interest in obtaining explicit formulae for $N$ or $\bar{\partial}^*N$ on special domains (see Phong [7] and Stanton [8] for the case of the Siegel domain $\mathbb{D}^n$).

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upper half space, and Harvey and Polking [1] for the ball). In particular, the representation for $\bar{\partial}^* N$ given here had been obtained first, by a different method, by Harvey and Polking [1]. In the meantime, Harvey and Polking extended their methods to obtain a representation for $N$ on the ball [2]. The work in [1] and [2] makes crucial use of the unitary invariance of the $\bar{\partial}$-Neumann problem on the ball, and thus it is not at all clear if and how those methods can be extended to more general domains. In contrast, as indicated above, the methods used here are not limited to the ball.

**Notation.** We freely use notations, terminology and results from [6]. The basic kernels and integral representation formulae for the case of the ball are briefly recalled in Sect. 3.

### 2. A Formula for $N$

Let $D$ be a relatively compact pseudoconvex domain with smooth boundary in a Hermitian manifold $X$. The complex Laplacian

$$\Box = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* : L^2_{0,q}(D) \to L^2_{0,q}(D)$$

with domain

$$\text{dom } \Box = \{ f \in L^2_{0,q} : f \in \text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^*, \bar{\partial} f \in \text{dom } \bar{\partial}^*, \bar{\partial}^* f \in \text{dom } \bar{\partial} \}$$

is a self-adjoint, densely defined linear operator. If the range $R(\Box)$ of $\Box$ on dom $\Box$ is closed, one has the orthogonal decomposition

$$L^2_{0,q}(D) = R(\Box) \oplus \ker \Box.$$  \tag{2.1}

One can then define the $\bar{\partial}$-Neumann operator $N = N_q : L^2_{0,q}(D) \to L^2_{0,q}(D)$, $0 \leq q \leq n$, by setting $N = 0$ on $\ker \Box$ and by defining $N f = u$ for $f \in R(\Box)$, where $u \in \text{dom } \Box$ is the unique solution of $\Box u = f$ which is orthogonal to $\ker \Box$. We denote by $H = H_q$ the orthogonal projection $L^2_{0,q}(D) \to \ker \Box$; note that $H_0$ is the Bergman projection. The following formulae are well known consequences of (2.1) – they hold for forms in the appropriate domains

\begin{align}
i) & \quad \Box N = N \Box = I - H \\
ii) & \quad HN = NH = 0 \tag{2.2} \\
iii) & \quad N \bar{\partial} = \bar{\partial} N, \quad \bar{\partial}^* N = N \bar{\partial}^*.
\end{align}

It is known that $R(\Box)$ is closed whenever the $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate. In this case, strong regularity results are known for $N$; in particular, $N$ is compact and pseudolocal. Subelliptic estimates hold in case $D$ is strictly pseudoconvex [4], and in case $bD$ is real analytic and does not contain any germs of complex subvarieties [5].

### 2.3. Theorem

Suppose $R(\Box)$ is closed and let

$$S_q = \bar{\partial}^* N_{q+1} : L^2_{0,q+1}(D) \to L^2_{0,q}(D).$$