A Table of Good Lattice Points in Three Dimensions*

Gershon Kedem and S. K. Zaremba

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Abstract. A table of three-dimensional lattice points with moduli ranging from 2120 to 6066, sufficient for high-precision numerical computation of triple integrals is presented; the data are shown to confirm some conjectures concerning good lattice points.

Good lattice points, to use the term introduced by Hlawka [1], provide us with a very efficient method of numerical integration over multidimensional intervals, or, more generally, over domains which can be conveniently reduced to multidimensional unit intervals (see, for instance, [1—3]). For the convenience of the reader, we recall the definition of good lattice points.

By lattice points in general we understand here points with integral coordinates; the latter will be denoted by a letter with subscripts running from 1 to s if s is the number of dimensions, while the same letter in bold face will denote the point itself. For any lattice point $h$ we put

$$R(h) = \max(1, |h_1|) \ldots \max(1, |h_s|).$$

Given an arbitrary positive integer $m$, which will be described as the modulus, and a lattice point $g$, let $\rho(g)$ be the minimum of $R(h)$ for all the lattice points $h \equiv 0 \equiv (0 \ldots 0)$ satisfying

$$g \cdot h \equiv 0 \pmod{m}, \quad (1)$$

where the dot denotes the scalar product. An $s$-dimensional lattice point $g$ is described as good modulo $m$ if

$$\rho(g) > (s-1)! m (2 \log m)^{1-s}.$$

The method of good lattice points for integrating a function $f(x)$ over the $s$-dimensional unit interval

$$Q^s: 0 \leq x_i \leq 1 \quad (i = 1, \ldots, s)$$

consists of taking as the approximate value of the integral the expression

$$\frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{k}{m} g\right),$$

where $g$ is a good lattice point modulo $m$, and the coordinates of the argument of $f$ are understood to be reduced modulo $m$. Upper bounds for the error in terms of $\rho(g)$ depend on the smoothness and, possibly, on the periodicity of $f$ [4].

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In particular, if \( f \) can be represented as a finite trigonometric polynomial

\[
f(x) = \sum_{R(h) < \theta(g)} c_h \exp(2\pi i h \cdot x),
\]

then the integration formula is exact. More generally, if: (i) for a positive integer \( r \) all the partial derivatives of \( f \) up to

\[
\frac{\partial^r f}{\partial x_1^r \ldots \partial x_s^r}
\]

are of bounded variation over \( Q_s \) in the sense of Hardy and Krause; and if: (ii) all the partial derivatives of \( f \) up to

\[
\frac{\partial^{(r-1)s} f}{\partial x_1^{r-1} \ldots \partial x_s^{r-1}}
\]

have values agreeing on opposite sides of \( Q_s \), then the error of integration has an upper bound of the form of 

\[
P^{(r+1)}(g) = \sum R(h)^{-(r+1)},
\]

the sum being extended to all \( h \neq 0 \) satisfying (1), and where \( K \) depends only on \( f \). The condition (ii) may appear unduly restrictive, but there are various methods of transforming the integral so as to satisfy this condition when (i) is satisfied [2, 4].

As to \( P^{(r+1)}(g) \), it has been proved [4] that for any sufficiently large \( m \) and any \( n \geq 2 \),

\[
P^{(n)}(g) \leq \frac{2^{3s+1} (\log m)^{s-1}}{(s-1)! (\log 2)^{s-1} \varrho(g)^n},
\]

but this is a rather crude upper bound; there is strong numerical evidence to support the conjecture that

\[
P^{(n)}(g) = O\left( \varrho(g)^{-n} \log m \right)
\]

irrespective of the number of dimensions.

If \( f \) does not satisfy the conditions (i) and (ii) above, but is of bounded variation in the sense of Hardy and Krause, and if \( g \) is a good lattice point, using the concept of discrepancy, one finds [4] that the error of integration is

\[
O\left( \frac{2^{3s-2} (\log m)^{2s-1}}{(s-1)! (\log 2)^{2s-1} m} \right),
\]

which is still much better than the accuracy claimed by the Monte Carlo method; this upper bound is also crude, and should be capable of improvement.

The existence of good lattice points with \( g_1 = 1 \) in any number \( s \geq 2 \) of dimensions modulo any sufficiently large \( m \) has been proved recently [5]; previously, a proof was known only for the case of a prime \( m \). It is conjectured, though, that for any \( s \geq 2 \), and any sufficiently large \( m \) there exist lattice points modulo \( m \) with \( \varrho(g) \) exactly of the order of \( m (\log m)^{2s-1} \). This is known to be true when \( s = 2 \), at least for some arbitrarily large \( m \), for if \( m = u_n g_1 = 1, g_2 = u_{n-1} \), where \( \langle u_i \rangle \) is the sequence of Fibonacci numbers, we have \( \varrho(g) = u_{n-2} \); in the case of \( s \geq 3 \), there is substantial numerical evidence to support this conjecture.