ESTIMATES OF THE SOLUTIONS OF TWO-POINT BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF FIRST-ORDER INTEGRODIFFERENTIAL EQUATIONS AND THE METHOD OF LINES

M. N. Yakovlev

One proves theorems on the estimates of the solutions of the systems of first-order integrodifferential equations

\[ \frac{d\mathbf{u}}{dt} = \sum_{j=1}^{n} \left[ a_{ij}(t)u_j(t) + \int_{0}^{T} k_{ij}(t,\tau)u_j(\tau)\,d\tau \right] + f_i(t), \quad 0 < t < T, \quad i = 1, \ldots, n, \]

with the boundary conditions

\[ \sum_{j=1}^{n} \left[ \alpha_{ij}u_j(0) + \beta_{ij}u_j(T) \right] = y_i, \quad i = 1, \ldots, n. \]

On the basis of these theorems, one suggests a method for estimating the norms of integrodifferential equations by the method of the lines for the solutions of the periodic boundary-value problems for second-order integrodifferential equations of parabolic type. On the basis of the established theorem on the solvability and on the estimate of the solution of the nonlinear equation

\[ T\mathbf{x} + F(x) = 0 \]

in a Banach space \( X \), where \( T \) is a linear unbounded operator, one investigates the convergence of the method of lines for solving the periodic boundary-value problem for a second-order nonlinear integrodifferential equation of parabolic type.

We give estimates for the solutions of systems of ordinary linear integrodifferential equations and inequalities of the first order with two-point boundary conditions. We give an existence theorem for the solution of a nonlinear operator equation (of the type of theorems on the modified Newton method) without the assumption regarding the boundedness of the derivative of the operator generating the equation. This, with the aid of the obtained estimates, allows us to give conditions for the convergence of the method of lines applied to nonlinear parabolic integrodifferential equations, without making use of the explicit estimates for the norm of the inverse operator of the linearized system of equations in the method of lines.

1°. For the vector-function \( \mathbf{u}(t) = \{ u_1(t), \ldots, u_n(t) \} \) we define the integrodifferential operator

\[ L_i \mathbf{u} = c_i(t) \frac{du_i}{dt} + \sum_{j=1}^{n} \left[ a_{ij}(t)u_j(t) + \int_{0}^{T} k_{ij}(t,\tau)u_j(\tau)\,d\tau \right] \quad i = 1, \ldots, n. \]  

Here the functions \( c_i(t), a_{ij}(t), k_{ij}(t,\tau), f_i(t) \) are continuous on the intervals considered below; \( c_1(t) \geq 0 \).

We consider the following system of ordinary linear integrodifferential inequalities

\[ \left[ L_i \mathbf{u} - f_i(t) \right] a_{ij}(t) > 0 \quad i = 1, \ldots, n; \]

\[ 0 < t < T \quad \text{for} \quad a_{ii}(t) > 0; \quad 0 < t < T \quad \text{for} \quad a_{ii}(t) < 0 \]

with the boundary conditions
\[ u_i(t) = \sum_{j=1}^{n} \left[ \alpha_{ij} \omega_j(0) + \beta_{ij} \omega_j(T) \right] \leq 0 \quad i = 1, \ldots, n. \] 

Assume that the vector-function \( \omega(t) = (\omega_1(t), \ldots, \omega_n(t)) \) is continuously differentiable and strictly positive (\( \omega_i(t) > 0 \)) for \( 0 \leq t \leq T \).

**Theorem 1.** Assume that \( \alpha_{ij} \leq 0, \beta_{ij} \leq 0 \) for \( i \neq j \) and that for each \( i \) one of the following conditions 1-4 holds.

1. \( \alpha_{ii} > 0, \beta_{ii} < 0, q_i = \frac{\sum_{j=1}^{n} \left[ \alpha_{ij} \omega_j(0) + \beta_{ij} \omega_j(T) \right]}{\omega_i(0)} > 0 \) and for \( 0 < t \leq T, 0 \leq \tau \leq T \) we have the inequalities
   \[ a_{ii}(t) < 0, K_{ij}(t, \tau) \geq 0, j = 1, \ldots, n; \quad a_{ij}(t) \geq 0 \quad \text{for} \quad i \neq j, \]

2. \( \alpha_{ii} \leq 0, \beta_{ii} > 0, q_i > 0 \) and for \( 0 \leq t < T, 0 \leq \tau \leq T \) we have the inequalities
   \[ a_{ii}(t) > 0, K_{ij}(t, \tau) \leq 0, j = 1, \ldots, n; \quad a_{ij}(t) \leq 0 \quad \text{for} \quad i \neq j. \]

3. \( \alpha_{ii} > 0, \beta_{ii} < 0, q_i \geq 0, \gamma_i = 0 \) and for \( 0 < t \leq T, 0 \leq \tau \leq T \) we have the inequalities
   \[ a_{ii}(t) < 0, K_{ij}(t, \tau) \leq 0, j = 1, \ldots, n; \quad a_{ij}(t) \geq 0 \quad \text{for} \quad i \neq j. \]

4. \( \alpha_{ii} < 0, \beta_{ii} > 0, q_i \geq 0, \gamma_i = 0 \) and for \( 0 < t < T, 0 \leq \tau \leq T \) we have the inequalities
   \[ a_{ii}(t) > 0, K_{ij}(t, \tau) \geq 0, j = 1, \ldots, n; \quad a_{ij}(t) \leq 0 \quad \text{for} \quad i \neq j. \]

Moreover, if \( d_{i0}(t_0) = 0 \), then there exists a sequence of indices \( i_0, i_1, \ldots, i_k \) and a sequence of numbers \( t_0, t_1, \ldots, t_k \) with \( d_{ik}(t_k) = 0 \) such that if \( d_{is}(t_s) = 0 \), then either \( a_{is}u_{s+1}(t_s) = 0 \) and \( t_{s+1} = t_s \) or \( K_{is}u_{s+1}(t_s, \tau) > 0 \) and \( t_{s+1} = t_s \).

Let \( \Omega_1 \) be the set of those \( t \) for which \( f_1(t)a_{ii}(t) \geq 0 \) and \( d_1(t) = 0 \). We assume that for \( t \notin \Omega_1 \) one has the inequality \( d_1(t) \neq 0 \).

Then, for the solution \( u(t) = (u_1(t), \ldots, u_n(t)) \) of the system of linear integrodifferential inequalities (2), subjected to the boundary conditions (3), one has the estimate
\[ u_i(t) \leq \omega_i(t) \max \left\{ 0, \max_{0 \leq s \leq T} \frac{\gamma_s}{q_i}, \max_{1 \leq s \leq \infty} \sup_{t \in I} \frac{f_s(t)}{-d_i(t)} \right\} \]
\[ i = 1, \ldots, n; \quad 0 \leq t \leq T. \quad (4) \]

Here \( I \) is either \((0, T)\) if for the considered \( i \) one has the inequality \( a_{ii}(t) < 0 \), or \([0, T)\) if for the considered \( i \) one has the inequality \( a_{ii}(t) > 0 \).

**Proof.** We set \( u_i(t) = \omega_i(t)s_i(t), i = 1, \ldots, n \). Then, for the vector-function \( S(t) = (S_1(t), \ldots, S_n(t)) \) we obtain the following system of inequalities
\[ \left[ c_i(t) \omega_i(t) - \sum_{j=1}^{n} \left[ a_{ij}(t) \omega_j(0) + \beta_{ij} \omega_j(T) \right] \right] - \frac{d}{dt} S_0(t) > 0 \quad i = 1, \ldots, n \quad (5) \]
with the boundary condition
\[ \sum_{j=1}^{n} \left[ a_{ij}(0) \omega_j(0) + \beta_{ij} \omega_j(T) \right] - \gamma_s \leq 0 \quad i = 1, \ldots, n. \quad (6) \]

Two cases are possible. Either \( S(t) \leq 0 \) for all \( 0 \leq t \leq T \) or for some \( i = i_0 \) and for some \( t = t_0 \), \( S_{i_0}(t_0) \) is the largest positive value, i.e.,
\[ S_{i_0}(t_0) = \max_{1 \leq s \leq \infty} \max_{0 \leq t \leq T} S_s(t) > 0. \quad (7) \]