Mass transport in flow of sources

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Abstract. Steady and unsteady local concentration has been determined analytically for two- and three-dimensional sources and is presented for various boundary-concentrations, volumetric flows and diffusion coefficients. The steady cases have been evaluated numerically. In addition an unsteady two-dimensional mass transport has been evaluated.

Stofftransport in Quellströmungen


Nomenclature

\( a \) \hspace{1cm} \text{inner radius of circle (2-dimensional case), inner radius of sphere (three-dimensional case)}

\( b > a \) \hspace{1cm} \text{outer radius of circle (2-dimensional case), outer radius of sphere (three-dimensional case)}

\( c \) \hspace{1cm} \text{concentration}

\( c_1, c_2 \) \hspace{1cm} \text{given concentration at the boundaries } r = a \text{ and } b \text{ resp.}

\( c_i \) \hspace{1cm} \text{initial concentration at the time } t = 0

\( D \) \hspace{1cm} \text{diffusion coefficient}

\( I_{k+1/2} \) \hspace{1cm} \text{modified spherical Bessel function}

\( J_{k+1/2} \) \hspace{1cm} \text{Bessel function of } k \text{-th order and first and second kind resp.}

\( k = \frac{b}{a} \) \hspace{1cm} \text{diameter ratio}

\( P_0^0(\xi) \) \hspace{1cm} \text{Legendre polynomials}

\( \rho, \varphi \) \hspace{1cm} \text{polar coordinates}

\( r, \theta, \varphi \) \hspace{1cm} \text{spherical coordinates}

\( t \) \hspace{1cm} \text{time}

\( u \) \hspace{1cm} \text{velocity in radial direction}

\( V_0 \) \hspace{1cm} \text{volumetric flow}

\( z_c = \frac{V_0}{4 \pi D} \) \hspace{1cm} \text{flow parameter for two-dimensional flow}

\( \beta_0 = \frac{V_0}{8 \pi D} \) \hspace{1cm} \text{flow parameter for three-dimensional flow}

\( \lambda_m \) \hspace{1cm} \text{eigenvalues}

\( \sigma_{\pm m} \) \hspace{1cm} \text{roots of determinant (28)}

\( \sigma = n^2 + z_c^2 \)

\( \xi = \cos \theta \)

\( \zeta = \frac{r}{a} \)

1 Introduction

Mass- and heat transfer problems in flowing media have observed quite some attention in recent years. The main area of research centered around the determination of the local temperature or concentration of a flowing liquid in conduits and tubes, in which either plug- or laminar flow was treated [1–6]. In all these investigations the cross-sectional geometry of the system was considered constant. The partial differential equation describing the problem could be separated and the remaining ordinary differential equation with mostly varying coefficients had to be solved by some numerical method, such as the Runge-Kutta procedure. In some cases exact analytical solutions could be obtained by proper transformation of the differential equation and yielded solutions with the help of the confluent hypergeometric function. For cases with changing cross-section, such as a diverging straight-lined conduit [7], conical tubes [8] or converging-diverging nozzles analytical [9] solutions could be found by the author. It also is, however, of interest to have solutions for liquid source flow for given boundary conditions and in the non-steady case also given initial local concentration at the time \( t = 0 \). Such studies are of interest to environmental engineers, which want to know the local concentration in a certain area. The following investigation treats the determination of the steady and non-steady local concentration for a two-dimensional and three-dimensional liquid sources, where two radial locations are kept at a given angular-dependent concentration, and where for the non-steady case the initial condition of the concentration field is given.

2 Basic equations and solutions

For the determination of the local concentration \( c \) in a moving liquid created by a two-dimensional or three-dimensional source at the origin, the second order differential equation

\[
D \left\{ \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c}{\partial \varphi^2} \right\} - \frac{V_0}{2 \pi} \frac{\partial c}{\partial r} = 0
\]

(1)
for the two-dimensional case, and

$$D \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial c}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial c}{\partial \theta} \right) \right\} - \frac{\dot{V}_0}{4 \pi r^2} \frac{\partial c}{\partial r} = 0$$

(2)

for the three-dimensional case has to be solved with the boundary conditions. In Eq. (1) \((\overline{r}, \phi)\) are the polar coordinates, \(\dot{V}_0\) is the volumetric flow, while Eq. (2) \((r, \theta)\) are the spherical coordinates. The value \(D\) is the diffusion coefficient.

### 2.1 Concentration for the two-dimensional source-flow

In the case of a two-dimensional source-flow in free space originating at the origin and exhibiting a velocity in radial direction \(u = \dot{V}_0/2\pi r\) the Eq. (1) has to be solved with the boundary condition

$$c = c_1(\phi) \text{ at the inlet } \overline{r} = a$$

(3)

and

$$c = c_2(\phi) \text{ at the location } \overline{r} = b > a.$$  

(4)

This means that the concentration at \(\overline{r} = a\) is kept at a magnitude \(c_1\), while that at \(\overline{r} = b\) is held at \(c_2\), which both could be functions of the angular coordinate \(\phi\).

The method of separation renders the solution of Eq. (1) as

$$c(\overline{r}, \phi) = \sum_{n=0}^{\infty} \left[ A_n \left( \frac{\overline{r}}{a} \right)^{-n} - B_n \left( \frac{\overline{r}}{a} \right)^n \right] \cos n\phi$$  

(5)

where

$$x_0 = -\frac{\dot{V}_0}{4 \pi D}$$

and where \(A_n\) and \(B_n\) are integration constants to be determined by the boundary conditions (3) and (4). It is

$$c_1(\phi) = \sum_{n=0}^{\infty} \left[ A_n + B_n \right] \cos n\phi \text{ and}$$

$$c_2(\phi) = \sum_{n=0}^{\infty} \left[ A_n k^{x_0 - \sqrt{x_0^2 + n^2}} + B_n k^{x_0 + \sqrt{x_0^2 + n^2}} \right] \cos n\phi$$  

(6)

where \(k = b/a > 1\).

Expanding the left-hand side of these equations into a Fourier-cosine series yields

$$A_n + B_n = \frac{1}{\pi} \int_0^{2\pi} c_1(\phi) \cos n\phi d\phi,$$  

(7a)

$$A_n k^{x_0 - \sqrt{x_0^2 + n^2}} + B_n k^{x_0 + \sqrt{x_0^2 + n^2}} = \frac{1}{\pi} \int_0^{2\pi} c_2(\phi) \cos n\phi d\phi$$  

(7b)

which renders the integration constants as

$$A_n = \frac{1}{\pi} \int_0^{2\pi} c_1(\phi) \cos n\phi d\phi - \frac{2\pi}{\pi} c_2(\phi) \cos n\phi d\phi$$  

(8a)

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} c_2(\phi) \cos n\phi d\phi - k^{-n} \frac{2\pi}{\pi} c_1(\phi) \cos n\phi d\phi$$  

(8b)

Introducing these Eq. (8a) in Eq. (5) yields the local concentration. If the boundary conditions show no angular dependence of \(\phi\), i.e. if \(c_1 = \text{const} \text{ and } c_2 = \text{const}\), the local concentration is presented for \(n = 0\) and is

$$c(\overline{r}) = \overline{A}_0 + B_0 \left( \frac{\overline{r}}{a} \right)^{2\pi x_0}$$

(9)

where

$$A_0 = (c_1 k^{2\pi x_0} - c_2)/(k^{2\pi x_0} - 1) \text{ and } B_0 = (c_2 - c_1)/(k^{2\pi x_0} - 1).$$

The local concentration is therefore for the axisymmetric case given by the expression

$$c(\overline{r}) = \frac{1}{(k^{2\pi x_0} - 1)} \left\{ (c_1 k^{2\pi x_0} - c_2) + (c_2 - c_1) (\frac{\overline{r}}{a})^{2\pi x_0} \right\}$$  

(10)

If the source strength ceases, i.e. the velocity \(u = \dot{V}_0/2\pi r\) vanishes, then \(x_0 = 0\) and we obtain, as may be seen from a limit consideration \(x_0 \to 0\) using the rule of l'Hospital the local concentration

$$c(\overline{r}) = c_1 + (c_2 - c_1) \ln \left( \frac{\overline{r}}{a} \right) \ln k.$$  

(11)

In the case of \(\phi\)-dependency it is

$$c(\overline{r}, \phi) = \frac{1}{2\pi} \int_0^{2\pi} c_1(\phi) d\phi$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \left[ c_2(\phi) - c_1(\phi) \right] d\phi \cdot \ln \left( \frac{\overline{r}}{a} \right) \ln k$$

$$+ \sum_{n=1}^{\infty} \left[ A_n \left( \frac{\overline{r}}{a} \right)^{-n} + B_n \left( \frac{\overline{r}}{a} \right)^n \right] \cos n\phi$$

(12)

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} \left[ k^{x_0} c_1(\phi) \cos n\phi d\phi $$

$$- k^{-n} \int_0^{2\pi} c_2(\phi) \cos n\phi d\phi \right] / (k^n - k^{-n})$$

(13a)

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} \left[ k^{x_0} c_2(\phi) \cos n\phi d\phi $$

$$- k^{-n} \int_0^{2\pi} c_1(\phi) \cos n\phi d\phi \right] / (k^n - k^{-n}).$$

(13b)

The influence of the flow velocity upon the local concentration is presented in Fig. 1 for the case of \(\phi\)-independent boundary conditions.