ON A CAUCHY PROBLEM FOR LAPLACE'S EQUATION

Sh. Yarmukhamedov

We obtain a formula which expresses the values of a harmonic function at points of a three-dimensional domain in terms of its values and the values of its normal derivative on a portion of the domain boundary.

1. Let \( x = (x_1, x_2, x_3) \), \( y = (y_1, y_2, y_3) \) be points of a three-dimensional Euclidean space \( \mathbb{E}^3 \), let \( D \) be a bounded domain in \( \mathbb{E}^3 \) with a piecewise-smooth boundary \( \partial D \), and let \( S_1 \) be a portion of \( \partial D \); finally, let \( S_2 = \partial D \setminus S_1 \). The problem (Cauchy problem) of determining the values of a harmonic function at points of a domain \( D \) from its values and the values of its normal derivative on the sub-boundary \( S_2 \) is an ill-posed problem in the classical sense (i.e., in the sense of Hadamard); however, if we narrow the class of functions considered (the class of functions correctly posed in the sense of Tikhonov [1]), the problem is no longer ill-posed in this new class. Having specified the Tikhonov class, we are naturally confronted with the problem of developing methods of effectively solving a given problem.

General methods for solving ill-posed problems for an operator equation were treated in [2-4]. We shall discuss the Cauchy problem for Laplace's equation in \( \mathbb{E}^3 \) in the setting described by Lavrent'ev in [2].

We say that a function \( \Phi(y, x, \delta) \) depending on a parameter \( \delta > 0 \) is a Carleman function for a domain \( D \) and the portion \( S_2 \) if, first of all, it is representable in the form

\[
\Phi(y, x, \delta) = r^{-1} + g(y, x, \delta),
\]

where \( g(y, x, \delta) \) is, everywhere in \( D \), a harmonic function in the variable \( y \), including the point \( y = x \), and is continuous in \( D \), and where \( r = |y - x| \); and if, secondly, the function \( \Phi \) satisfies the inequality

\[
\frac{1}{4\pi} \int_{S_1} \left( |\Phi| + \left| \frac{\partial}{\partial n} \Phi \right| \right) ds_y \leq \delta.
\]

Here, \( ds_y \) is an area element of the surface \( S_1 \), and \( \partial/\partial n \) indicates differentiation along the exterior normal to \( S_1 \).

Existence of the Carleman function implies stability of the solution of the Cauchy problem; an effective method of constructing this solution is equivalent to formulating an effective method of solving the problem approximately when the Cauchy boundary data on \( S_2 \) is specified approximately [2].

In the case of two dimensions the solution of the Cauchy problem for Laplace's equation is equivalent [2, 5] to the problem of determining the values of a holomorphic function at the points of a two-dimensional domain from a knowledge of its boundary values on a part of the boundary. Since holomorphic functions form a ring, construction of the Carleman function is somewhat simpler here [2].

The case of harmonic functions of three variables is considerably more involved, since this property is lacking here. When \( D \) is an arbitrary simply connected domain with a smooth boundary \( \partial D \), and \( S_2 \) is an arbitrary piece with a smooth edge, existence of the Carleman function was established in [6] by Mergelyan. The method given in [6] is a very complicated one. A much simpler method for constructing the Carleman function for a subset of the domain \( D \) was proposed in [2].
2. Let $G$ be a bounded domain (not necessarily simply connected), bounded by the plane $y_3 = 0$ and by smooth surfaces lying in the half-space $y_3 > 0$.

We construct the Carleman function for this domain when the Cauchy data are specified on $S_2 = \partial G / S_1$, where $S_1$ is the planar portion of the boundary $\partial G$, $y_3 = 0$.

We put

$$\alpha^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2.$$ 

For $\alpha > 0$ we define the function $\Phi(y, x, \sigma)$ by the expression

$$\Phi(y, x, \sigma) = \frac{2}{\pi} e^{-\alpha} \Re \left[ \exp \left\{ \frac{\sigma (\sqrt{y^2 + z^2 + i y_3})}{i \sigma} \right\} \frac{du}{\sqrt{y^2 + \alpha^2 + i (y_3 - z)}} \frac{1}{\sqrt{y^2 + \alpha^2}} \right]_{x_3 > 0, \sigma > 0}. \tag{1}$$

**Lemma.** The function $\Phi(y, x, \sigma)$, defined for $\alpha > 0$ by expression (1), is representable in the form

$$\Phi(y, x, \sigma) = \frac{1}{r} + g(y, x, \sigma),$$

where $r = |y - x|$, and where $g(y, x, \sigma)$ is a function defined for all values of $y$ and $x$ and harmonic in $y$ throughout the space $E^3$.

**Proof.** For convenience we introduce the notation

$$f(z + iy_3) = \frac{\exp \left\{ \frac{\sigma (z + iy_3)}{i \sigma} \right\}}{\sqrt{z + i (y_3 - z)}},$$

$$q(y, x) = \int_0^\infty \frac{f(Vu^2 + \sigma^2 + i y_3)}{\sqrt{Vu^2 + \sigma^2}} \frac{du}{Vu^2 + \sigma^2}.$$ 

In this notation we have

$$\Phi(y, x, \sigma) = 2/\pi e^{-\alpha} \Re q(y, x).$$

Our aim here is to prove that the function $\phi(y, x)$ is harmonic in $y$ for $\alpha > 0$ (from which it will then follow that the function $\Phi$ will also be harmonic in $y$ for $\alpha > 0$).

Differentiating the function $\phi(y, x)$, we obtain

$$\frac{\partial \phi}{\partial y_k} = \int_0^\infty \frac{y_k - x_k}{u^2 + \alpha^2} \left( f' - \frac{f}{\sqrt{u^2 + \alpha^2}} \right) du,$$

$$\frac{\partial^2 \phi}{\partial y_k \partial y_l} = \int_0^\infty \frac{3(y_k - x_k)^2 - u^2 - \alpha^2}{(u^2 + \alpha^2)^2 \sqrt{u^2 + \alpha^2}} f du + \int_0^\infty \frac{u^2 + \alpha^2 - 3(y_k - x_k)^2}{(u^2 + \alpha^2)^2} f' du + \int_0^\infty \frac{(y_k - x_k)^2 f'' du}{(u^2 + \alpha^2)^3}.$$ 

$$\frac{\partial^2 \phi}{\partial y_k \partial y_l} = -\int_0^\infty \frac{f'' du}{\sqrt{u^2 + \alpha^2}},$$

$$\Delta \phi = \int_0^\infty \frac{(x^2 - 3 u^2) f du}{(u^2 + \alpha^2)^2 \sqrt{u^2 + \alpha^2}} + \int_0^\infty \frac{2 u^2 - x^2}{(u^2 + \alpha^2)^2} f' du + \int_0^\infty \frac{u^2 f'' du}{(u^2 + \alpha^2)^3 \sqrt{u^2 + \alpha^2}}.$$ 

Integrating the third integral by parts, we find that it is equal to

$$-\int_0^\infty \frac{\alpha^2 - x^2}{(u^2 + \alpha^2)^3} f' du.$$