On the lattice of manifolds of all algebras $\mathcal{L}$ we study the operator of nilpotent closure $J: \alpha \rightarrow \alpha + \mathfrak{F}$, where $\mathfrak{F}$ is a nilpotent manifold of $\mathfrak{M}$-algebras. With a given system of identities $\Sigma$ defining $\mathfrak{M}$, we construct a system $\Sigma^*$, giving the manifold $\alpha + \mathfrak{F}$. It is proved that if $\alpha$ does not contain $\mathfrak{M}$, then the lattice of submanifolds of $\alpha + \mathfrak{F}$ is the double of the lattice of submanifolds of $\alpha$. We describe the free and subdirect indecomposable manifolds of algebras $\alpha + \mathfrak{F}$. Let $B \subseteq \alpha + \mathfrak{F}$ and $A$ be a dense retract of $B$. We denote by $\theta(B)$ the lattice of congruences on $B$. The theorem is proved: $\theta(B)$ is a complemented lattice if and only if $\theta(A)$ is a complemented lattice.

1. For any domain of operators $\Omega$ which does not contain operators with no arguments, we denote by $\mathcal{L}\Omega$ the lattice of all manifolds of $\Omega$-algebras. We consider the manifold $\mathfrak{M}$, defined by the system of identities $\Sigma_0 = \omega_1(x_1, \ldots, x_n) = \omega_2(y_1, \ldots, y_m)$, where $\omega_1, \omega_2 \in \Omega$. One calls $\mathfrak{M}$ a nilpotent manifold of signature $\Omega$. It is clear that $\mathfrak{M}$ is an atom in the lattice $\mathcal{L}\Omega$. Our goal: to study on the lattice $\mathcal{L}\Omega$ the map $J_{\mathfrak{M}}: \gamma \rightarrow \gamma + \mathfrak{M}$. More precisely, to study the algebra structure from the manifold $\gamma + \mathfrak{M}$ for arbitrary $\gamma \in \mathcal{L}\Omega$.

Clearly the map $J_{\mathfrak{M}}$ is a closure operator on the lattice $\mathcal{L}\Omega$.

Let $\alpha$ be an arbitrary manifold and $\Sigma$ be some system of identities. Let us arrange to denote by $\alpha'$ the collection of all identities which are satisfied on all algebras from $\alpha$, and by $\Sigma'$ the manifold defined by the system $\Sigma$. We consider two arbitrary manifolds $\alpha$ and $\beta$, for which $\alpha' \subseteq \beta'$ and $\beta' \subseteq \alpha'$. We denote by $[\Sigma]$ the collection of all identities deducible from $\Sigma$ and not deducible from $\Sigma_0$, i.e., $[\Sigma] = \Sigma' \setminus \Sigma'$. 

**Proposition 1.** If $\alpha + \mathfrak{M} = \beta + \mathfrak{M}$ and $[\alpha'] \cap [\beta'] = \emptyset$, then $\alpha = \beta$.

**Proof.** Let $\alpha + \mathfrak{M} = \beta + \mathfrak{M}$ and $[\alpha'] \cap [\beta'] = \emptyset$; then one can find an identity of the form $u(x) = x$ belonging to both $\alpha'$ and $\beta'$. We assume that for some system of identities $\Sigma$ we have: $[\Sigma] = \emptyset$. We fix some identity of the form $w(x) = x$, deducible from $\Sigma$, and we consider an arbitrary identity $p = t$ from $[\Sigma]$. Changing each variable $y$ in this identity into $w(y)$, we get a new identity, deducible from $\Sigma$. From this it follows that the system $\Sigma$ can equivalently be represented in the form $\Sigma_{\Omega} \cup \{u(x) = x\}$, where $\Sigma_{\Omega}$ consists only of identities belonging to $\Sigma_0$. We assume now that $\Sigma$ gives the manifold $\alpha + \mathfrak{M}$. But then, by virtue of the facts established above, the system $\Sigma \cup \{u(x) = x\}$ is equivalent to the system $\alpha'$, i.e., $\alpha' = \beta'$.

**Corollary 1.** On the lattice of manifolds of semigroups the operator $J_{\mathfrak{M}}$ will be a one-one map.

In fact, for manifolds of semigroups $\alpha$ and $\beta$, which are understood not to contain $\mathfrak{M}$, always $[\alpha] \cap [\beta] = \emptyset$.

**Corollary 2.** Any system of identities $\Sigma$ can be represented equivalently in the form $\Sigma_{\Omega} \cup \{u(x) = x\}$, where $u(x) = x \in \Sigma$ and $\Sigma_{\Omega} = \Sigma'$.

We shall go on now to elucidate the following question: how are the algebras from $J_{\mathfrak{M}}(\gamma)$ arranged for preassigned manifold $\gamma$? We introduce the following definition. Let $A$ and $B$ be two arbitrary $\Omega$-algebras. One will call $B$ an $\mathfrak{M}$-extension of the algebra $A$ if: i) $A$ is isomorphic to the algebra $\Omega(B) = \bigcup \omega(B, \ldots, B)$;
ii) there exists an endomorphism $\varphi$ of the algebra $B$, which is the identity on the subalgebra $\Omega(B)$; i.e., $\Omega(B)$ is a retract of the algebra $B$. 

The complete description of the algebras from the manifold $J_{\varnothing}(\alpha)$ is given by the following

**THEOREM 1.** The algebra $B$ belongs to the manifold $J_{\varnothing}(\alpha)$ if and only if it is an $\mathfrak{R}$-extension of some algebra of the manifold $\alpha$.

**Proof.** Let $B \in J_{\varnothing}(\alpha)$ and $B \not\in \alpha$. From this it follows that on $B$ no identity from $[\alpha']$ can be satisfied. We consider on $B$ the partition defined by the classes $\Omega(B)$ and $B \setminus \Omega(B)$. It is clear that $\Omega(B)$ is a subalgebra of $B$, while it belongs to $\alpha$. Let $u(x) = x \in [\alpha']$. We define on $B$ a map $\varphi$, by setting $\varphi(x) = u(x)$. Clearly $\varphi$ is an endomorphism of $B$ which is the identity on $\Omega(B)$.

We assume now that some algebra $B$ is an $\mathfrak{R}$-extension of an algebra $A$ of the manifold $\alpha$. We choose an arbitrary identity $u = v \in \alpha' \cap [\alpha']$ and we shall show that it is satisfied on $B$. In fact, if this identity is not trivial, then $u = \varphi(u) = \varphi(v) = v$. Thus it is proved that $B \in J_{\varnothing}(\alpha)$. The theorem is completely proved.

We proceed now to the presentation of a general method for finding for a given system of identities $\Sigma$ a system of identities $\Sigma^*$ defining the nilpotent closure $\Sigma'$.

**LEMMA.** If the manifold $\alpha$ is given by the identity $u(x) = x$, then its nilpotent closure is defined by the system $\Sigma_u = \{\omega(x_1, ..., x_n) = \omega(u(x_1), x_2, ..., x_n), ..., \omega(x_1, ..., x_n) = \omega(x_1, ..., x_{n-1}, u(x_n))\}$, $u(\omega(x_1, ..., x_n)) = \omega(x_1, ..., x_n)$, where $\omega$ is an arbitrary operator from $\mathfrak{R}$.

**Proof.** It is clear that the identity $u(x) = x$ derives from the system $\Sigma_u$. We consider an algebra $B$ from the manifold $\Sigma_u$ and we shall show that it is an $\mathfrak{R}$-extension of some algebra satisfying the identity $u(x) = x$, whence on the basis of Theorem 1 it will follow that $B \in J_{\varnothing}(\alpha)$. The theorem is completely proved.

Let $A = \Omega(B)$; then obviously $A \in \alpha$. We consider on $B$ the map $\varphi: x \mapsto u(x)$. Since on $B$ the identities of the system $\Sigma_u$ are satisfied,

$$\varphi(\omega(x_1, ..., x_n)) = \omega(\varphi(x_1), ..., \varphi(x_n)) = \omega(x_1, ..., x_n),$$

and thus everything is proved.

We consider an arbitrary system of identities $\Sigma$, but such that $\Sigma \not\subset [\alpha']$. By virtue of Corollary 2, $\Sigma$ can be represented in the form $\Sigma_u \cup \{u(x) = x\}$.

**THEOREM 2.** The system of identities $\Sigma^* = \Sigma_u \cup \Sigma_u$ gives the nilpotent closure of the manifold $\Sigma'$.

In fact, on the basis of the lemma and Theorem 1 it follows that each algebra from $\Sigma^*$ is an $\mathfrak{R}$-extension of some algebra $\Sigma'$, and hence $\Sigma^*$ defines the manifold $J_{\varnothing}(\Sigma')$.

**COROLLARY 1.** The operator $J_{\varnothing}$ is an endomorphism on the lattice $\mathfrak{L}_{\Omega}$. If $\alpha, \beta \in L_{\Omega}$, then always $J_{\varnothing}(\alpha + \beta) = J_{\varnothing}(\alpha) \cap J_{\varnothing}(\beta)$. We assume that $\alpha' \not\subset [\alpha']$ and $\beta' \not\subset [\beta']$, because if $\alpha' \subset [\alpha']$ and $\beta' \subset [\beta']$, then there is nothing to prove. We have

$$J_{\varnothing}(u(x) = y) = J_{\varnothing}(y(x) = y).$$

By virtue of Corollary 2 of Theorem 1, $\alpha'$ and $\beta'$ are representable respectively in the form $\alpha' = \cup \{u(x) = x\}$ and $\beta' = \cup \{v(x) = x\}$, and using the lemma, we have:

$$J_{\varnothing}(\alpha' \cup \beta') = J_{\varnothing}(\alpha' \cup \beta') = J_{\varnothing}(\alpha' \cup \beta') \cap J_{\varnothing}(\beta').$$

Now let $\alpha' \not\subset [\alpha']$ and $\beta' \not\subset [\beta']$. Applying an analogous argument, we also get $J_{\varnothing}(\alpha' \cap \beta') = J_{\varnothing}(\alpha') \cap J_{\varnothing}(\beta')$. For any manifold $\gamma$ we denote by $\mathfrak{R}_{\gamma}$ the lattice of all submanifolds of $\gamma$. Let $L$ be some lattice and $L_1$ a sublattice of it. If $L \setminus L_1 = L_2$ is also a sublattice of $L$ and there exists an isomorphism $f: L_1 \to L_2$ such that $x \equiv f(x)$, then one says that $L$ is the double of $L_1$.

**COROLLARY 2.** If $\alpha' \not\subset [\alpha']$, then the lattice $J_{\varnothing}(\alpha)$ is the double of the lattice $\mathfrak{R}_{\alpha}$.

In fact, we consider the map $J_{\varnothing}: \mathfrak{R}_{\alpha} \to J_{\varnothing}(\alpha)$. Since any two elements of $\mathfrak{R}_{\alpha}$ satisfy the condition $[\gamma'] \cap [\beta'] = \phi$, by virtue of Proposition 1 the map $J_{\varnothing}$ will be one-one and by Corollary 1 an isomorphism of the lattices $\mathfrak{R}_{\alpha}$ and $L_1 = J_{\varnothing}(\alpha) \setminus L_2$. But since always $\gamma \subset J_{\varnothing}(\gamma)$, everything is proved.

We now consider some examples of nilpotent closures of concrete manifolds of algebras.