Some formulations of a number of inverse problems for potentials of various types of equations are considered. In the first part, the problems of existence, uniqueness, and stability of solutions of inverse problems for generalized potentials of equations of elliptic type are studied. In the second part, some formulations of a number of inverse problems for other types of equations are given, in particular, for wave and heat potentials, and a number of inverse problems for the transport equation are also posed.

Inverse problems of potential theory are problems in which it is required to find the shape and density of an attracting body on the basis of given values of the exterior (interior) potential of this body or on the basis of given values of functionals of the potentials in question. In another formulation, one such problem consists in finding a body such that its "exterior" volume potential of given density should coincide outside this body with a given harmonic function. Problems of existence, uniqueness, stability, and also the development of "effective" numerical methods of solution occupy a central position in the investigation of the problems indicated. As concerns the problem of existence, at the present time there are no criteria for a global solution of these problems. There are a number of theorems for existence in the small in so-called inverse problems for a body close to a given body, but even in this case there are considerable difficulties in the study of the equations, which, are, in general, nonlinear, to which these problems reduce. Therefore, in many cases the existence of global solutions of these problems is assumed a priori (this is a natural assumption for many applied problems), and the problems of uniqueness and stability are investigated. Many inverse problems have a nonunique solution. Therefore, one of the principal tasks in studying the problem of uniqueness is to discover additional conditions which solutions should satisfy in order that they be unique. The problem of stability of the problems indicated is closely related to the problem of uniqueness. For those problems which are written in the form of an equation of first kind it is the case, in general, that arbitrarily small variations of the right side can correspond to finite variations of the solutions; i.e., these problems belong to the incorrectly posed problems of mathematical physics. In order that the problem become correct, it is necessary to impose a number of additional restrictions on the solutions; with these restrictions different characteristics of the deviation of the solution in dependence on the deviation of the right side are obtained. The applied importance of inverse potential problems is so great that they have recently been counted among the current problems of modern mathematical analysis. Of fundamental importance in this direction are the investigations of A. N. Tikhonov [21, 22], P. S. Novikov [8], L. N. Sretenskii [20], I. M. Rapoport [19], V. K. Ivanov [1, 2], and M. M. Lavrent’ev [5]. In a series of works of the author [10–18] inverse problems have been studied for classical and generalized potentials of elliptic equations of second order; all the results proved in these works are new even for classical potentials, while the problems are considered in space and, in the majority of cases, directly
for generalized potentials which then include the classical potentials for the Laplace equation as a special case.

In presenting the general formulations of inverse problems for elliptic potentials we shall restrict ourselves to the volume potential and the potential of a simple layer for the Laplace equation in three-dimensional Euclidean space, although the problems indicated have been studied in n-dimensional Euclidean space (n \geq 2) for potentials of general elliptic equations.

Let $T_\alpha (\alpha = 1, 2)$ be simply connected, bounded domains with piecewise smooth boundaries $S_\alpha$. We introduce the volume (Newtonian) mass potential and the potential of a simple layer

\begin{align*}
  u_\alpha (x) &= u(x; T_\alpha, \mu_\alpha) = \frac{1}{|x-y|} \mu_\alpha (y) dy; \\
  v_\alpha (x) &= v(x; S_\alpha, \zeta_\alpha) = \frac{1}{|x-y|} \zeta_\alpha (y) dS_\alpha,
\end{align*}

where $|x-y|$ is the distance between the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $E^3$, $\mu_\alpha (y) \neq 0 ( \zeta_\alpha (y) \neq 0)$ almost everywhere in $T_\alpha (S_\alpha)$. We set $Z_\alpha (x) = \beta u_\alpha (x) + \gamma v_\alpha (x)$, where $\beta, \gamma$ are real numbers, $\beta^2 + \gamma^2 \neq 0$.

1. The general exterior inverse potential problem consists in finding the shape and densities of the attracting body on the basis of given values of the exterior potential $Z_\alpha (x)$. For the uniqueness of the solution of this problem we present another formulation of it.

Problem 1. Find conditions for the domains $T_\alpha$ and the densities $\mu_\alpha, \zeta_\alpha$, such that equality of the exterior potentials $Z_1(x) = Z_2(x)$, $Z_1(x) = Z_2(x)$

\begin{equation}
  Z_1(x) = Z_2(x) \quad \text{for} \quad x \in E^3 \setminus (T_1 \cup T_2),
\end{equation}

should imply the equalities $T_1 = T_2, \mu_1 = \mu_2, \zeta_1 = \zeta_2$. We point out that if the set $E^3 \setminus (T_1 \cup T_2)$ consists of a single component, then condition (1) is satisfied when $Z_1(x) = Z_2(x)$ for $|x| = R$, where $R > 0$ is sufficiently large or when data are given on the boundary of the sphere $|x| = R$ which ensure that $Z_1(x)$ and $Z_2(x)$ coincide outside the sphere. Examples of such data are Dirichlet data on the entire boundary of the sphere or Cauchy data on a piece of the boundary of the sphere, etc.

It is possible to give a functional formulation of Problem 1. We have the following

**LEMMA.** In order that $Z_\alpha (x)$ satisfy condition (1) it is necessary and sufficient that for the functionals

\begin{equation}
  J_\alpha (h) = \beta \int_{S_\alpha} \mu_\alpha (y) h(y) dy + \gamma \int_{S_\alpha} \zeta_\alpha (y) h(y) dS_\alpha,
\end{equation}

the equality $J_1(h) = J_2(h)$ hold, where $h(y)$ is a function which is harmonic in the domain $D \supset (T_1 \cup T_2)$. In other words, in order that the exterior magnetic potentials be equal it is necessary and sufficient that their harmonic moments be equal.

Thus, the general exterior inverse problem consists in finding the shape and densities of the attracting body on the basis of the values of the harmonic moments. We note that if the boundaries $S_\alpha$ of the domains $T_\alpha$ are not smooth, then the question of equality of the functionals $J_1(h)$ and $J_2(h)$ for harmonic functions in the domain $T_\alpha$ forming the interior of $T_1 \cup T_2$, $h \in L_p(T_\alpha)$ ($p \geq 1$), is equivalent to the problem of approximation in $L_p(T_\alpha)$-spaces of harmonic functions by harmonic polynomials [3]. For example, there is an example due to M. V. Keldysh and M. M. Lavrent'ev where a function harmonic in a simply connected domain and continuous in its closure cannot be uniformly approximated by harmonic polynomials in the closure of this domain.

In the sequel we shall assume for simplicity that $S_\alpha \subset A^{(\mu, \lambda)}$, or are piecewise smooth, while the sets $T' = T_1 \cap T_2$ and $T'' = E^3 \setminus (T_1 \cup T_2)$ consist of a single component. We note that for the general exterior inverse problem the solution is unique if $\mu_\alpha (y) = \mu (y) > 0, \zeta_\alpha (y) = \zeta (y) > 0, \beta, \gamma > 0$, and the domains $T_\alpha$ are contact domains, i.e., such that for each of the sets $T'$ and $T''$ there exists a common portion $S_\ast (\text{mes} S_\ast = 0)$ of the boundaries $S_\alpha$ with $\text{mes} (S_1 \cup S_2) \setminus S_\ast = 0$.

We shall consider the general Problem 1 for the potential of volume mass, i.e., in (1) $\beta = 1, \gamma = 0$. This problem we shall call the mixed exterior inverse problem for determining the shape and density of the attracting body on the basis of the values of the external potential.