GENERALIZATION OF THE PROBLEM OF THE NUMBER
OF PARTITIONS OF A FINITE SET

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A unified scheme is proposed for obtaining the generating functions for combinatorial objects defined on partitions of finite sets.

A number of expressions are well known in enumerative combinatorial mathematics for generating functions having the same essential nature. Among them are the generating function for the number of partitions of a finite set [2], the relations between the generating functions of all possible graphs of a particular type with labeled vertices and connected graphs of the same type, etc. The purpose of the present note is to show that all of these facts are closely affiliated with the enumeration theory of Polya [1] and are special cases of a single general result, which we formulate and prove below.

Let us fix a certain countable set \( M \). We shall tacitly denote finite subsets of \( M \) by the symbol \( D \), attaching indices or primes as needed. We also denote by \( I_k \) for every natural \( k \) the set \( \{1, 2, \ldots, k\} \). The notation \( |D| \) indicates the number of elements in \( D \). Let there correspond to every nonempty set \( D \) a certain (possibly empty) finite set of objects \( S_D \), which we call elementary structures on \( D \). We shall assume that if \( D \) and \( D' \) are nonempty sets such that

\[
D \cap D' = \emptyset,
\]

then also

\[
S_D \cap S_{D'} = \emptyset.
\]

We fix a certain set \( D \) and consider the set

\[
F_D^{(k)} = \bigcup_{D_1, D_2, \ldots, D_k} S_{D_1} \prod_{i \in I_k} S_{D_i},
\]

in which \( k > 0 \) is an integer, the union is taken over the possible \( k \)-partitions of \( D \) (i.e., over partitions of \( D \) into \( k \) subsets), and \( \prod_{i \in I_k} S_{D_i} \) is the product of the family sets \( S_{D_i}, i \in I_k \) [5]. We call the elements \( f \in F_D^{(k)} \) \( k \)-structures on \( D \). It is understood that \( F_D^{(k)} = \emptyset \) if \( k > |D| \). Every element \( f \in F_D^{(k)} \) describes a mapping of the set \( I_k \) into \( \bigcup_{D' \subseteq D} S_{D'} \), which is the product of mappings: \( f = \varphi \psi \), where \( \varphi \) places in correspondence with the element \( i \in I_k \) a subset \( D_i \subseteq D \), where \( D_1, D_2, \ldots, D_k \) form an ordered \( k \)-partition of \( D \) and \( \varphi(D_i) \) defines an elementary structure \( S_i \in S_{D_i} \).

Let \( C \) be a commutative ring containing the rational numbers, \( m \geq 0 \) an integer, and \( C[[x_i]]_{i \in I_{m+1}} \) the algebra of formal power series of the variables \( x_i, i \in I_{m+1} \), with coefficients from \( C \) [6]. We denote the variable \( x_{m+1} \) simply by \( x \). If \( u \in C[[x_i]]_{i \in I_{m+1}} \) is a formal series without a free term, then the family of series \( (1/n!)u^n, n = 0, 1, 2, \ldots, \) where \( u^0 = 1 \), is summable. We denote their sum by \( e^u = \exp \{u\} \).

We assume that there is assigned to every elementary structure a weight \( w(s) \), i.e., an element of the algebra \( C[x_i]_{i \in I_m} \); for \( m = 0 \) the weight \( w(s) \in C \). If \( f \in F_D^{(k)} \) and \( f(i) = s_i, i \in I_k \), we set \( W(f) = \sum_{i=1}^{k} w(s_i) \).

We say, by definition, that \( w(S_D) = 0 \) if \( S_D = \emptyset \) and that \( w(S_D) = \sum_{i \in D} w(s(i)) \) otherwise. For every \( k = 1, 2, \ldots \), let a certain substitution group \( G_k \) act on the set \( I_k \). This group induces an equivalence relation on the sets \( F_D^{(k)} \): two \( k \)-structures \( f_1, f_2 \in F_D^{(k)} \) are said to be equivalent if there is a substitution \( g \in G_k \) such that \( f_1(g(i)) = f_2(i) \) for all \( i \in I_k \). It is easily shown that this equivalence is indeed an equivalence relation and can be used to partition \( F_D^{(k)} \) into equivalence classes. If two \( k \)-structures from \( F_D^{(k)} \) are equivalent, they have the same weight, a fact that is proved as in [1], Sec. 5, and it is admissible to define this common value as the weight of the equivalence class. If \( \mathcal{F} \) is an equivalence class, we denote its weight by \( W(\mathcal{F}) \).

We now find \( \sum_{\mathcal{F}} W(\mathcal{F}) \), where the summation extends over all classes \( \mathcal{F} \) into which the set \( F_D^{(k)} \) is partitioned. For this we note that every class \( \mathcal{F} \) comprises \( |G_k| \) distinct \( k \)-structures, where \( |G_k| \) is the order of the group \( G_k \). This result ensues from the fact that the components of every \( \mathcal{F} \)-structure are distinct elementary structures. We therefore have

\[
\sum_{\mathcal{F}} W(\mathcal{F}) = \frac{1}{|G_k|} \sum_{f \in F_D^{(k)}} W(f),
\]

and by the definitions and notation introduced above relation (1) can be rewritten as follows:

\[
\sum_{\mathcal{F}} W(\mathcal{F}) = \frac{k!}{|G_k|} \sum_{D_1 \cup D_2 \cup \ldots \cup D_k = D} \prod_{i=1}^{k} w(S_{D_i}),
\]

where the sum is taken over distinct \( k \)-partitions of \( D \).

**Theorem.** For every natural \( k \) let the group \( G_k \) be a symmetric group of degree \( k \), and every time that \( |D| = |D'| \) let the equality \( w(S_D) = w(S_{D'}) \) hold. Let us put \( w|D| = w(S_D) \) and

\[
W_D = 1, \quad W_{|D|} = \sum_{k=1}^{|D|} \sum_{\mathcal{F}} W(\mathcal{F}),
\]

where \( D \neq \emptyset \) and the summation on \( \mathcal{F} \) in (3) is taken over all equivalence classes of the set \( F_D^{(k)} \). Then the following relation is valid in the algebra \( C[x_i]_{i \in I_{m+1}} \):

\[
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} = \exp \left( \sum_{n=1}^{\infty} w_n \frac{x^n}{n!} \right).
\]

**Proof.** It follows from the second premise of the theorem that the quantities \( w|D| \) and \( W|D| \) do not depend on the choice of the set \( D \), but depend only on its power; for every \( n \geq 1 \) the quantities \( W_n \) and \( w_n \) in (4) denote \( W|D| \) and \( w|D| \), respectively, where \( D \) is an arbitrary set of power \( n \). Making use of (2) and the fact that \( |G_k| = k! \), \( k = 1, 2, \ldots \), we obtain

\[
W_n = \sum_{k=1}^{n} \sum_{D_1 \cup D_2 \cup \ldots \cup D_k = D} \prod_{i=1}^{k} w|D_i|,
\]

where the meaning of the inner sum in (5) has been explained above. We order all possible partitions of \( D \) into \( k \) parts. We pick an arbitrary set of nonnegative integers \( k_1, k_2, \ldots, k_n \) satisfying the conditions

\[
k_1 + k_2 + \ldots + k_n = k, \quad k \cdot k_1 + 2 \cdot k_2 + \ldots + n k_n = n,
\]

i.e., the set \( k_1, k_2, \ldots, k_n \) is a partition of the number \( n \) into \( k \) parts, and we consider the partition of \( D \) into those \( k \) subsets \( D_1, D_2, \ldots, D_n \) from which the \( k_i \) contain \( i \) elements, \( i \in I_n \). We know (from, e.g., [2]) that the number of such partitions of a set of power \( n \) is equal to

\[
\frac{n!}{(1)!^k (2)!^k \ldots (n)!^k k_1! k_2! \ldots k_n!},
\]