SELF-NORMALIZING NILPOTENT SUBGROUPS OF THE FULL LINEAR GROUP OVER A FINITE FIELD

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It has been proved (Ref. Zh. Mat., 1977, 4A170) that in the full linear group \( \text{GL}(n_2, q) \), over a finite field of \( q \) elements, \( q \) odd or \( q = 2 \), the only self-normalizing nilpotent subgroups are the normalizers of Sylow 2-subgroups and that for even \( q > 2 \) there are no such subgroups. In the present note it is deduced from results of D. A. Suprunenko and R. F. Apatenok (Ref. Zh. Mat., 1960, 13586; 1962, 9A150) that this is true for any \( n \).

It was proved in [5] that in any finite solvable group there exist self-normalizing nilpotent subgroups (Carter subgroups) and that all such subgroups are conjugate. It is of interest to study nonsolvable groups in which there are Carter subgroups. A well-known result of P. Hall asserts that the Sylow 2-subgroups of the symmetric groups are self-normalizing. Carter subgroups of \( \text{GL}(n_2, q) \), the full linear group of degree \( n \) over a finite field \( F_q \) of \( q \) elements, were studied in [4]. It was shown that if \( n = 2 \) or \( n = 3 \), the following is true: If \( q \) is odd or \( q = 2 \), there exists a single conjugacy class of Carter subgroups in \( \text{GL}(2, q) \) and \( \text{GL}(3, q) \), namely, the normalizers of the Sylow 2-subgroups; if \( q \) is even and \( q > 2 \), then there are no Carter subgroups in these groups (the case \( n = 2, q = 2^{n+1} \) was considered earlier in [8]). In [4] it was asked whether these results can be extended to the case of the full linear group of arbitrary degree \( n \). In the present paper we show that an affirmative answer to this question is an almost direct consequence of the results of [2] and [3], which are devoted to a description of the maximal irreducible nilpotent subgroups of \( \text{GL}(n_2, q) \). More precisely, we have the following.

**Theorem.** In the full linear group \( G = \text{GL}(n, q) \) of degree \( n > 2 \) over a field \( F_q \) of \( q \) elements, \( q \) odd or \( q = 2 \), there exists a single conjugacy class of self-normalizing nilpotent subgroups, namely, the normalizers of the Sylow 2-subgroups. If \( q > 2 \) is given, \( G \) contains no self-normalizing nilpotent subgroups.
1. Sylow Subgroups

As usual, we denote by $\text{Syl}_p(G)$ some Sylow $p$-subgroup of the finite group $G$, by $Z(G)$ the center of $G$, and by $[X]$ the integer part of the number $X$. In the sequel, $\ell$ is a rational prime different from $p$ and from 2. If $m$ is a natural number, we write $\ell^r|m$ if $\ell^r|\ell^n$ and $\ell^r|\ell^m$.

**Lemma 1.** Suppose $G = \text{GL}(n, q)$, $q = p^k$, $n \geq 2$, $\ell$ is a prime different from 2 and from $p$. Then the group $H = Z(G)\text{Syl}_\ell(G)$ is different from its normalizer in $G$.

**Proof.** Recall how the Sylow $\ell$-subgroups of $G$ are constructed (see [3] or [7], Theorem 1.4B). Suppose $e$ is the smallest natural number such that $\ell^e|\ell^q - 1$. Suppose first that $n = \ell^e$. A Sylow $\ell$-subgroup $P_\ell$ of $\text{GL}(\ell^e)$ has order $\ell^{h_i}$, where $h_i = \ell^e + (1 + \ell + \ldots + \ell^{e-1})$. Its structure is as follows: $P_\ell$ is a cyclic group of order $\ell^r$, and $P_i = P_{i-1} \otimes Z_\ell$ is the wreath product of $P_{i-1}$ and a cyclic group $Z_\ell$ of order $\ell$, where in the role of $Z_\ell$ we can choose, for example, the subgroup generated by the matrix

$$
\begin{pmatrix}
0 & E & 0 \\
E & 0 & \ddots \\
& & & 0
\end{pmatrix}
$$

where $E$ is the identity matrix of order $\ell^{e-1}$. It is clear that if $i > 0$, then $P_i$ is normalized, for example, by the matrix

$$
\begin{pmatrix}
0 & E & 0 \\
E & 0 & \ddots \\
& & & 0
\end{pmatrix}
$$

not belonging to $Z(G)P_\ell$. We now consider the realization of the group $P_\ell$. Suppose $\eta$ is a generator of the multiplicative group of the field $\mathbb{F}_q$, $\eta$ the minimal polynomial of $\eta$ over $\mathbb{F}_q$. Suppose $f_\ell(x) = \alpha_0 + \alpha_1 x + \ldots + x^e$, where $\alpha_0, \ldots, \alpha_{e-1} \in \mathbb{F}_q$. Then the matrix

$$
A = \begin{pmatrix}
0 & & -\alpha_0 \\
& \ddots & \ddots & \ddots \\
& & & & \alpha_0 \\
0 & & & & 1 - \alpha_{e-1}
\end{pmatrix}
$$

is conjugate over $\mathbb{F}_q$ to the matrix $\text{diag}(\eta, \eta^q, \ldots, \eta^{q^{e-1}})$ and therefore has order $q^{e-1}$. The matrix $A^{(q^{e-1}/\ell^e)}$ generates $P_\ell$. But since, by hypothesis, $\ell \not| 2$, the equality $\ell^r = q^{e-1}$ is impossible and therefore the subgroup $P_\ell$ is properly contained in the subgroup generated by $A$, and the group $Z(G)P_\ell$ is not self-normalizing in $\text{GL}(\ell, q)$ if $e > 1$.

Now suppose $n$ is arbitrary. Put $d = \lfloor n/e \rfloor$ and represent $d$ in the form $d = d_0 + d_1 \ell + \ldots + d_k \ell^k$, $0 \leq d_i < \ell$. Then a Sylow $\ell$-subgroup $P$ of $\text{GL}(n, q)$ has the form $I x_1 \times \ldots \times P_{i_0} \times \ldots \times P_{i_k}$, where $i$ occurs $n - d e$ times, $P_0$ occurs $d_0$ times, ..., and $P_k$ occurs $d_k$ times. By what has already been proved, $N_\ell(Z(G)P) = Z(G)P$, if $d_i > 0$ for some $i > 0$, or if $e > 1$ and $d_0 > 0$. But in the remaining cases $e \geq n$ and $e = 1$, $\ell > n$, the subgroup $P$ is normalized by all monomial matrices.