In this paper we present the necessary and sufficient condition of epimorphism of the operator
\[ \mathcal{F} : H^\mu(R^n) \ni u(x) \rightarrow \{Q_j(D)u,...,Q_m(D)u\}_{j=1}^{m} \in H^\mu(R^m), \]
where the \( Q_j(d) \) are differential operators with constant coefficients, \( R^m \) is a subspace of \( R^n \), and \( H^\mu(R^n) \) and \( H^\mu(R^m) \) are distribution spaces introduced in [1]. We prove the existence of a linear continuous operator \( \pi \) which is the right inverse of \( \mathcal{F} \). There are 4 references.

In [1], spaces \( H^\mu(R^n) \) were introduced and studied in detail that consist of distributions \( u \in S' \) whose Fourier transforms \( \hat{u}(\xi) \) have a square integrable (with weight \( \mu^2(\xi) \)) absolute value.* The class of spaces \( H^\mu(R^n) \) plays an important role in the theory of partial differential operators. This is due to the fact that it is closed under the action of any such operator \( Q(D) \) (in any case if the latter has constant coefficients), and also to the fact that this class contains among its representatives the spaces \( W^{(2)}_2 \ldots 2^{(n)} \) of Sobolev-Slobodetskii and the spaces \( H^Q \).

In [2], which deals with the regularity properties of distributions that lie in the domains of definition of differential operators, the problem was studied of the traces in a subspace \( R^m \subset R^n \) (consisting of the points \( x = (x', 0), x' = (x_1, \ldots, x_m) \)) of distributions \( Q(D)u \), where \( u \in H^\mu(R^n) \), and \( Q(D) \) is an arbitrary (scalar or matrix) differential operator with constant coefficients. It was found (if we confine ourselves to the scalar case) that either the function \( \nu(\xi) = (Q_1(\xi)/|Q_1(\xi)|)^{1/n}, \xi = (\xi_1, \ldots, \xi_m) \), vanishes identically and then the operator \( \mathcal{F} : \Omega \ni u(x) \rightarrow \{Q(D)u\}_{j=1}^{m} \in S'(R^m) \) cannot have a closure for any domain \( \Omega \subset R^n \), or \( \nu(\xi) \neq 0 \) for all \( \xi \not\in R^m \), in which case this operator can be extended to a continuous operator \( \mathcal{F} : H^\mu(R^n) \rightarrow H^\mu(R^m) \), where \( H^\mu(R^m) \) is the narrowest of the spaces of the class under consideration that contains traces \( \{Q(D)u(x', 0)\}_{R^m} \) for all elements \( u \in H^\mu(R^n) \). But the following problems remained unsolved: firstly, to ascertain the conditions under which the traces in \( R^m \) of elements \( Q(D)u(x) \) for \( u \in H^\mu(R^n) \) fill entirely the space \( H^\mu(R^m) \), and secondly, if the latter is true, does there exist a continuous extension operator \( \pi : H^\mu(R^m) \rightarrow H^\mu(R^n) \) such that \( \mathcal{F} \) is an identity mapping?

In the present paper we give a complete answer to these questions (in a framework, much more general than the one just considered) both in the case of scalar spaces \( H^\mu(R^n) \) and in the case of vector spaces and matrix differential operators \( Q(D) \). Let us note that for some particular cases our results are contained in [3] and [4].

1. Notation and Preliminary Details

Let us denote by \( R^n \) the \( n \)-dimensional space of points \( x = (x_1, \ldots, x_n) \) or \( \xi = (\xi_1, \ldots, \xi_n) \), where all the \( x_i \) and \( \xi_i \) are real numbers. By \( Q(\xi) = Q(\xi_1, \ldots, \xi_n) \) we shall denote a polynomial of the vector \( \xi \), and by \( Q(D) \) the corresponding differential operator obtained from \( Q(\xi) \) by replacing \( \xi_j \) by \(-i\partial/\partial x_j \). Moreover, \( Q(\xi) = (\Sigma Q^{(\alpha)}(\xi)^{1/n}, \) where \( Q^{(\alpha)}(\xi) \) is a mixed derivative of \( Q(\xi) \) of order \( \alpha = (\alpha_1, \ldots, \alpha_n) \), the sum being taken over all possible values of \( \alpha \) (which are finitely many). The symbol \( \prod_{i=1}^{n} H_i \) denotes the direct product of the spaces \( H_1, \ldots, H_f \) with the ordinary topology of pointwise convergence. \( S' \) will denote

*All the necessary definitions are given in Section 1.
Schwartz's space of slowly increasing distributions. If \( u(x) \in \mathcal{S}' \), then \( \hat{u}(\xi) \) will denote the Fourier transform of \( u(x) \). We shall also write

\[ \hat{u}(\xi) = F_{\kappa \rightarrow \xi} u(x) \text{ and } u(x) = F^{-1}_{\xi \rightarrow \kappa} \hat{u}(\xi). \]

By definition the class \( \mathcal{S}_{(0)} \) of weight functions consists of those and only those functions \( \mu(\xi) \) that are continuous and satisfy for all \( \xi, \eta \in \mathbb{R}^n \) the inequality

\[ \frac{\mu(\xi)\mu(\eta)}{\xi + \eta} \leq c (1 + |\xi - \eta|)^l \]

for some constant \( c > 0 \) and \( l \). \( \mathcal{S}_{(0)} \) is a group under multiplication and for any polynomial \( Q(\xi) \) we have

\[ \hat{Q}(\xi) \in \mathcal{S}_{(0)} \quad (B, p. 9). \]

By definition the space \( H^p(\mathbb{R}^n) \) consists of distributions \( u \in \mathcal{S}' \) such that the integral

\[ \|u\|^p = \int \mu^p(\xi)|\hat{u}(\xi)|^p\,d\xi \]

is finite. Since \( \mu(\xi) > 0 \), any of the spaces \( H^p(\mathbb{R}^n) \) will be isometrically isomorphic to the space \( L_2(\mathbb{R}^n) \).

This isomorphism can be established with the aid of the mapping

\[ I_u: H^p(\mathbb{R}^n) \ni u(x) \mapsto \mu(\xi)\hat{u}(\xi) \in L_2(\mathbb{R}^n), \]

whose inverse is

\[ I_u^{-1}: L_2(\mathbb{R}^n) \ni v(\xi) \mapsto F_{\xi \rightarrow \kappa}^- \mu^{-1}(\xi)v(\xi). \]

If \( m < n \), we shall use for the points \( x \in \mathbb{R}^n \) the notation \( x = (x^t, x^m) \), where \( x^t = (x_1, \ldots, x_m) \in \mathbb{R}^m \), \( x^m = (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m} \) and accordingly \( \xi = (\xi^t, \xi^m) \). If \( Q(D) \) is a differential operator, we shall write instead of \( \left[(Q(D)u)\right]_{\mathbb{R}^m} \) the quantity \( Q(D)u(x^t, 0) \).

2. Some Auxiliary Propositions.

We shall denote by \( \mathcal{R}^l \) a complex Euclidean space whose elements are the numerical vectors \( v = (v_1, \ldots, v_l) \), \( w = (w_1, \ldots, w_l) \) etc. (all the vector quantities are printed in boldface) with complex components. By definition, the scalar product of two vectors \( v \) and \( w \) in \( \mathcal{R}^l \) and the norm of the vector \( v \) are expressed by

\[ [v, w] = \sum_{i=1}^{l} v_i\overline{w_i}, \quad |v| = \left(\sum_{i=1}^{l} |v_i|^2\right)^{1/2} \]

(the bar denotes the complex conjugate quantity). If \( G \) is a matrix of order \( l \) consisting of the numbers \( G_{ij} \), its determinant will be denoted by \( \det G \) and its trace by \( \text{Tr} G \).

A family of matrices \( G(t) \) is said to be uniformly positive definite if for some positive constant \( c \) and all \( t \) and any \( v \in \mathcal{R}^l \) we have the inequality

\[ [G(t)v, v] \geq c |v|^2. \]

**PROPOSITION 1.** In order that a family of positive definite Hermitian matrices \( G(t) \) be uniformly positive definite it is necessary, and if \( \text{Tr} G(t) \leq M < \infty \), then it is also sufficient, that

\[ \det G(t) \geq c > 0, \quad (2) \]

where \( c \) is a positive constant, not dependent on \( t \).

**Proof.** Let \( 0 < \lambda_1(t) \leq \ldots \leq \lambda_l(t) \) be the eigenvalues of the matrix \( G(t) \). Since for any \( t \) we have

\[ \inf |G(t)v, v|/|v|^2 = \lambda_1(t), \]

where the infimum is taken over all \( v \in \mathcal{R}^l \), it follows from (1) that \( \lambda_1(t) \geq c \) for all \( t \), and since \( \text{det} G(t) = \lambda_1(t) \ldots \lambda_l(t) \), it follows that \( \text{det} G(t) \geq c^l \). Conversely, from inequality (2) and from the fact that \( \sum_{i=1}^{l} \lambda_i(t) \leq M < \infty \), it follows directly (taking into account the positiveness of all \( \lambda_i(t) \)) that \( \lambda_1(t) \geq c > 0 \) for some constant \( c \) which does not depend on \( t \); using once more the fact that \( \inf |G(t)v, v|/|v|^2 = \lambda_1(t) \), we hence obtain the inequality (1) with \( c = c' \).

Let us recall that if \( H \) is a complex Hilbert space with a scalar product \( \langle \cdot, \cdot \rangle \) and \( e_1, \ldots, e_l \) are any of its elements, then the Gram matrix \( G(e_1, \ldots, e_l) \) will be defined as a Hermitian matrix formed by the numbers \( G_{ij} = \langle e_i, e_j \rangle \). Below we shall utilize the following proposition.