A DISTRIBUTION FUNCTION WHOSE MOMENTS ARE CLOSE TO THOSE OF A NORMAL LAW

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The distance in the uniform metric between two distribution functions one of which is normal is considered in terms of the distances between their corresponding moments (in some order). This result is applied to bounds on the distance in the uniform metric of two distribution functions, the moments of whose convolutions are close to the corresponding moments (in some order) of a normal law, from the set of all normal distribution functions. Three references are given.

It is well known [1] that if the moments of a distribution tend to the moments of a normal law, then the corresponding distribution function tends to the corresponding normal distribution function. At the same time, bounds which would permit us to judge the rate of convergence of the distribution functions in terms of the rate of convergence of the moments are of interest. In this note, besides giving bounds of this kind in Theorem 2, we mention an application to the problem of the stability of a theorem due to H. Kramer about the normality of the components of a normal law. These results are based on the following theorem of N. A. Sapogov [2].

**THEOREM 1.** Let

\[ F = F_1 * F_2 \]

be a convolution of distribution functions, where \( F_1(x) \) has zero median and

\[ \sup_x |F(x) - \Phi(x)| \leq \delta < 1, \]

where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{t^2}{2} \right\} dt. \]

We set

\[ R = \sqrt{-2 \ln \delta} + 1, \]

\[ a_i = \int_{-R}^{R} x dF_i(x), \quad b_i^2 = \int_{-R}^{R} x^2 dF_i(x) - a_i^2 \quad (i = 1, 2). \]

Then there is an absolute constant \( C \) such that if \( b_1^2 > 0 \) then

\[ \sup_x \left| F_i(x) - \Phi_i \left( \frac{x - a_i}{b_i^2} \right) \right| \leq \frac{C}{b_i^2 \sqrt{-\ln \delta}} \quad (i = 1, 2). \]

**THEOREM 2.** Let the moments of order \( 2n \geq 4 \) of the distribution function \( V(x) \) be finite and satisfy the relations

\[
\begin{align*}
\int x \, dV(x) &= a, \\
\int (x - a)^2 \, dV(x) &= \sigma^2 > 0, \\
\int (x - a)^{2k-1} \, dV(x) &= (2k - 2)!! \sigma^{2k-1} \sqrt{\pi} \times z^k (2k - 1) \delta_{2k-1}, \\
\int (x - a)^{2k} \, dV(x) &= (2k - 1)!! \sigma^{2k} (1 + \pi 2^{k+1} k \delta_{2k}), \\
k &= 2, 3, \ldots, n.
\end{align*}
\]

Set

\[ \varepsilon_m = \frac{2^m}{m \pi} + \sum_{k=1}^{m} |\delta_k| \quad (2 \leq m \leq n). \]
Then there is an absolute constant $C$ such that if
\[ \varepsilon = \min_{m} \varepsilon_m < 1, \] (6)
then
\[ \sup_x \left| V(x) - \Phi \left( \frac{x - \mu}{\sigma} \right) \right| \leq \frac{C}{\sqrt{-2 \ln \varepsilon}}. \] (7)

**Proof.** It is sufficient to prove the existence of the constant $C$ in (7) for a fixed pair of numbers $\alpha$, $\sigma > 0$. We assume
\[ \alpha = 0, \quad \sigma^2 = \sigma_0^2 = \frac{1}{\alpha}. \] (8)
We set
\[ \beta_k = \max_x \left| q^{(k)}(x) \right| \quad (k = 0, 1, 2, \ldots). \] (9)
Let us consider the convolution
\[ U(x) = \int \Phi \left( \frac{x - y}{\sqrt{1 - \sigma_0^2}} \right) dV(y). \] (10)
We will denote by $\theta$ a number of absolute value not greater than unity and by $C_*$ an absolute constant.

For any integer $m \in [2, n]$ we have
\[ U(x) = \int dV(y) \left[ \sum_{k=0}^{m-1} \frac{\phi^{(k)}(\theta^2 \sigma_0^2)}{k!} \mu^k + \frac{\Theta_{m-1}}{2m!} \mu^{2m} \right], \]
where
\[ \nu = \frac{x}{\sqrt{1 - \sigma_0^2}}, \quad \mu = -\frac{\theta}{\sqrt{1 - \sigma_0^2}}. \]
In view of (4) we may write
\[ U(x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(\theta^2 \sigma_0^2)}{k!} \mu^k + \frac{\Theta_{m-1}}{2m!} \mu^{2m} - \sqrt{2\pi} \sum_{k=0}^{m-1} \frac{\phi^{(k-1)}(\theta^2 \sigma_0^2)}{(2k-3)!} \beta_{k-1} + \frac{\pi}{\sqrt{2\pi}} \sum_{k=2}^{m-1} \frac{\phi^{(k)}(\theta^2 \sigma_0^2)}{(2k-2)!} \beta_k + \frac{\pi}{\sqrt{2\pi}} \sum_{k=2}^{m} \frac{\phi^{(k)}(\theta^2 \sigma_0^2)}{(2k-3)!} \beta_{k-1}. \] (11)
Analogously, by considering the convolution
\[ \Phi(x) = \int \Phi \left( \frac{x - y}{\sqrt{1 - \sigma_0^2}} \right) d\Phi \left( \frac{y}{\sigma_0^2} \right), \]
we obtain
\[ \Phi(x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(\theta^2 \sigma_0^2)}{k!} \theta^k + \frac{\Theta_{m-1}}{2m!} \theta^{2m}. \] (12)
From (11) and (12), we learn that
\[ \beta_{k-1} \ll \frac{(2k-3)!}{\sqrt{2\pi}}, \quad \beta_k \ll \frac{(2k-2)!}{\pi} \quad (k = 2, 3, \ldots), \]
and we obtain
\[ \sup_x \left| U(x) - \Phi(x) \right| \ll \varepsilon. \] (13)
We set
\[ T = \sqrt{-2 \ln \varepsilon} + 1, \]
\[ a = \int_{-T}^T x dV(x), \quad b^* = \int_{-T}^T x^2 dV(x) - a^2. \]