MINIMAL SURFACE AS A DIAGRAM OF SPHERE ROTATIONS

I. Kh. Sabitov

Based on I. N. Vekya's representation of the field of infinitely small (i. s.) bendings of a sphere in terms of analytic functions, we present a new proof of Liebmann's theorem to the effect that the diagram of rotations of i.s. bendings of a sphere is a minimal surface and, conversely, each minimal surface is the diagram of rotations of some i.s. bending of a sphere or of part of it. It is then established that all the minimal surfaces which are non-trivially locally isometric to a given minimal surface constitute an analytic single-parameter family, and explicit expressions for the surfaces of this family are given. The bibliography contains four titles.

1. Let \( z \) be the field of non-trivial infinitely small bendings of some portion of the unit sphere with radius \( r \), let \( y \) be its corresponding field of rotations, while \( s = z - \{yr\} \) is the field of translations (for the definitions, see, e.g., N. V. Efimov's paper [1]). By laying out all the \( y \) and \( s \) from a single point, we obtain, respectively, certain surfaces, \( Y \) (the diagram of the rotations) and \( S \) (the diagram of displacements). Liebmann [2] had already obtained the following local results: 1) he proved that \( Y \) is a minimal surface, 2) he showed (but with insufficient foundation) that, for each regular minimal surface \( X \), there exists an infinitely small bending of it such that the corresponding diagram of rotations is a region on a sphere, and 3) he verified that this infinitely small bending takes the minimal surface, \( X \), into an infinitely close minimal surface, \( X_e \). Since each surface is a diagram of rotations for its own diagram of rotations, Liebmann's Theorem can be formulated as follows: Each diagram of rotations of a sphere is a minimal surface, and, conversely, each minimal surface is the diagram of rotations of a sphere. In section 2 of this paper, we shall give a proof in terms of analytic functions of a more complete theorem (than Liebmann's) namely: 1) we shall again prove Liebmann's Theorem as we have just stated it; 2) we shall prove the uniqueness of the infinitely small bending of the given minimal surface, \( Y \), for which the diagram of rotations is a spherical region, and we shall convince ourselves that it coincides with field \( s \); 3) we shall verify that the diagram of translations, \( S \), is also a minimal surface.

In part 3 we shall study the question of the set of minimal surfaces which are non-trivially locally isometric to a given minimal surface. It turns out that such surfaces always exist and, in their totality, comprise a one-parameter analytic family such that each of them is obtained from another by a continuous and finite bending (see, Theorem 2). In particular, \( Y \) and \( S \) belong to this family.

In order to establish a complete one-to-one correspondence between minimal surfaces and diagrams of sphere rotations, it is necessary for us, on the one hand, to admit into consideration non-single-valued (with branch points) and unbounded infinitely small bendings and, on the other hand, to consider generalized minimal surfaces, i.e., minimal surfaces which can have isolated singularities in the form of branch points. In this case, all the assertions made earlier are true *on the whole* if one takes into account the analyticity of the generalized minimal surfaces.

For the sake of brevity, we shall always henceforth replace *infinitely small bending* and *generalized minimal surface* by, respectively, *bending* and *minimal surface*.

2. Thus, we agree that we shall consider bendings which are either single-valued or non-single-valued (which can be considered single-valued for the corresponding multi-sheeted regions), and which are either bounded or unbounded (in this latter class, all the spheres are bendable).

**THEOREM 1.** The rotations diagram, \( Y \), of any bending of a sphere or part thereof is a minimal surface. Conversely, any minimal surface is the diagram of rotations of some bending of a sphere or part thereof.

**Proof.** Before anything else, we note, first, that both surfaces, the diagram of rotations and the minimal surface, are analytic and, second, that their possible singularities are isolated. It is therefore
incumbent on us to prove the coincidence of both surfaces in a singly-connected neighborhood of some regular point.

Let \( Y \) be the diagram of rotations. On the unit sphere we introduce an isometrically conjugate coordinate system \((\xi, \eta)\), assuming that \( \xi, \eta \) are the stereographic coordinates of points of the sphere when they are projected from the south pole onto the plane of the equator. In his book [3], I. N. Vekua gives the formulas (with the condition that the xy plane coincides with the equatorial plane):

\[
r(\xi, \eta) = \left( \frac{-2\xi}{1 + \xi^2}, \frac{2\eta}{1 + \xi^2}, \frac{1 - \xi^2}{1 + \xi^2} \right) = n(\xi, \eta).
\]

\[
z(\xi, \eta) = \text{Im} \left( \Phi(\xi) (n_\xi - i n_\eta) - \left[ \Phi'(\xi) - \frac{2\xi}{1 + \xi^2} \Phi(\xi) \right] n \right),
\]

where \( \xi = \xi + i\eta, r(\xi, \eta) \) is the radius vector of the sphere, \( n(\xi, \eta) \) is its unit external normal, \( z(\xi, \eta) \) is the field of bending of the region under consideration, and \( \Phi(\xi) \) is an analytic function. If \( r^a r_c = \delta^a_c \) (\( a, c = 1, 2 \)), \( z = u_+ r^a + u_0 n \) is the bending, then (see [3])

\[
(u_+ + i u_0)(1 + \xi^2) = 4 \Phi(\xi).
\]

It is clear from (2) and (3) that these formulas establish a one-to-one correspondence between the bendings of the sphere (or its parts) and analytic functions, it being easily calculated that the bending will be trivial for functions of the form \( \Phi(\xi) = a + \xi^2 + \xi^2 \) and only for them.

From the definition of field \( y \), we have \( ds = |y dr| \). We set \( \frac{\partial}{\partial \xi} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \), \( \frac{\partial}{\partial \eta} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \), and we shall seek \( y \) in the form \( y = A(\xi) r_\xi + A(\eta) r_\eta + vn \). With (1), (2) and the definition of \( y \) taken into account, we can obtain the following by direct computation:

\[
A(\xi) = \frac{i}{2} \left[ \Phi(\xi) + \xi \Phi'(\xi) \right] - \frac{i(1 + \xi^2)}{4} \left[ \frac{2 \Phi'(\xi)}{1 + \xi^2} - \Phi''(\xi) \right],
\]

\[
v(\xi) = \text{Im} \left[ \Phi'(\xi) + \frac{2 \Phi'(\xi)}{1 + \xi^2} \right].
\]

Now, after simple, but extensive, computations, we obtain

\[
y = A r_\xi + A r_\eta + vn = (y_1, y_2, y_3),
\]

\[
y_1(\xi) = \text{Im} \left[ -\Phi(\xi) + \xi \Phi'(\xi) + \frac{1 - \xi^2}{2} \Phi''(\xi) \right],
\]

\[
y_2(\xi) = \text{Re} \left[ \Phi(\xi) + \frac{1 + \xi^2}{2} \Phi''(\xi) \right],
\]

\[
y_3(\xi) = \text{Im} \left[ \Phi'(\xi) - \frac{1}{2} \Phi''(\xi) \right].
\]

It is known that in isometrically conjugate coordinate systems we have

\[
y_1 = ax + \beta r_\xi, \quad y_2 = \beta r_\eta - ax, \quad y_3 = v.
\]

Since \( r_\xi^2 = r_\eta^2 = \frac{4}{(1 + \xi^2)^2} \), \( r_\xi r_\eta = 0 \), we thus easily obtain that

\[
y_1^2 = y_3^2 = \frac{4(\alpha^2 + \beta^2)}{(1 + \xi^2)^2}, \quad y_1 y_3 = 0.
\]

Consequently, the coordinate system \((\xi, \eta)\), for the \( Y \) neighborhood in question, is conformal and, since all the components of \( y \) are, according to (4), harmonic functions, \( Y \) then coincides with some minimal surface.

If we now compute \( \alpha, \beta \), as well as the coefficients \( L, M, N \) of the second quadratic form of surface \( Y \), we find

\[
L = -N = \text{Im} \Phi''(\xi), \quad M = \text{Re} \Phi''(\xi),
\]

\[
4\alpha = (1 + \xi^2)^2 \text{Im} \Phi''(\xi), \quad 4\beta = (1 + \xi^2)^2 \text{Re} \Phi''(\xi).
\]