ON CONJUGATE FUNCTIONS AND TRIGONOMETRIC SERIES

L. V. Zhizhiashvili

The present paper is an abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Physical and Mathematical Sciences. The dissertation was defended on June 30, 1967 before the Academic Council of the Faculty of Mechanics and Mathematics of the M. V. Lomonosov Moscow State University. The official opponents were Professor A. V. Efimov, Doctor of Physical and Mathematical Sciences, Professor V. A. Il'in, Doctor of Physical and Mathematical Sciences, and Professor G. P. Tolstov, Doctor of Physical and Mathematical Sciences.

This paper investigates some problems of the theory of Fourier series and the trigonometric series conjugate to them, examining both simple series and multiple (chiefly double) series. The study consists of four chapters.

The first chapter discusses some properties of the conjugate functions

\[ \tilde{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x + t) \cot \frac{t}{2} \, dt \]

and the Hilbert transformations

\[ \tilde{f}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} \, dt \]

for functions of one variable. Thus, for the conjugate functions, the chapter contains proofs of theorems which are generalizations of the well-known inequality of A. N. Kolmogorov [1]. For example, it is established that for a sufficiently broad class of functions \( \omega(x) \) the relation

\[ \left( \int_{-\infty}^{\infty} \left| \tilde{f}(x) \omega(x) \right|^p \, dx \right)^{1/p} \]

\[ \leq A(p, \omega) \int_{-\infty}^{\infty} |f(x)| \, dx \int_{-\infty}^{\infty} |\omega(x)| \, dx, \quad 0 < p < 1 \]  \( \ldots \) (1)

is satisfied, where \( A(p, \omega) \) is a positive constant dependent on the parameters indicated. Further, it is proved that for sufficiently arbitrary functions \( \omega(x) \) and sufficiently smooth positive functions \( \varphi(t) \) the condition

\[ \int_{-\infty}^{\infty} |f(x + h + \eta) - f(x + \eta)| \omega(x) \, dx \]

\[ = O\{\varphi(h)\}, \quad (|h|, |\eta|) \leq \lambda_0, \quad 1 \leq p < +\infty \]

implies

\[ \int_{-\infty}^{\infty} |\tilde{f}(x + h + \eta) - \tilde{f}(x + \eta)| \omega_0(x) \, dx \]

\[ = O\{\varphi_1(h)\}, \quad (|h|, |\eta|) \leq \lambda_0', \quad 1 \leq p < +\infty, \]

where \( \varphi(h) \sim |h|^{-\gamma} \), and \( \lambda_0, \lambda_0' \) are fixed positive numbers. This assertion is a generalization of a well-known result of Hardy and Littlewood [2] which states that if \( f(x) \in \text{Lip}(a, p), 0 < a < 1, \quad 1 \leq p < +\infty, \) then it is also true that \( \tilde{f}(x) \in \text{Lip}(a, p) \).

The properties of the Hilbert transformation are studied next; an analog of Eq. (1) is established for the case of the function \( \tilde{f}(x) \). In particular, for even functions \( f(x) \in L(-\infty, +\infty) \) the inequality

\[ \left( \int_{-\infty}^{\infty} \left| \frac{\tilde{f}(x) \omega(x)}{1 + x^2} \right|^p \, dx \right)^{1/p} \]

\[ \leq A_1(p, \omega) \int_{-\infty}^{\infty} \frac{|f(x)|}{1 + x^2} \, dx \int_{-\infty}^{\infty} \frac{\omega(x)}{1 + x^2} \, dx, \quad 0 < p < 1. \]

will hold. The problem of the summability of the function $\tilde{f}(x)$ is studied in detail. The conclusions reached along this line show that in a number of cases the functions $\tilde{f}(x)$ and $\check{f}(x)$ have substantially different properties.

The second chapter deals with the problems of convergence and summability of Fourier series and of the trigonometric series conjugate to them; it discusses the properties of conjugate functions of even and odd functions. Thus, the chapter contains proofs of theorems which specify sufficient conditions for $(C, \alpha > -1)$-summability (at isolated points, almost everywhere, or uniformly within some interval $[a, b] \cap [-\pi, \pi]$) of Fourier series and of the trigonometric series conjugate to them. For these same series the chapter discusses problems related to the approximation properties of their Cesàro means. Specifically along this line, estimates are found for the quantity $\int |c_n^s(x) - f(x)|^p dx, \; a > -1, \; 1 \leq p < +\infty \; \|L^\infty \| = C$ by using the integral modulus of continuity of the function $f(x)$, where $c_n^s(x)$ are the Cesàro means of a Fourier series of the function $f(x)$. An analogous problem is studied for the conjugate trigonometric function as well. In particular, from the conclusions reached here the corresponding results of Hardy and Littlewood [2], K. Iano [3], M. Sato [4], P. L. Ull'yanov [5], and others are derived.

This chapter also contains an examination of some properties of the functions

\[
\psi(x) \equiv \psi(x, f) = \frac{x}{2} \int_0^x f(t) \, dt,
\]

\[
\psi(x) \equiv \psi(x, f) = \int_0^x f(t) \, \text{ctg} \frac{t}{2} \, dt,
\]

\[
0 < x < \pi,
\]

when the original function $f(x)$ is even or odd;* problems related to the behavior of partial sums of the Fourier series of the functions $\varphi(x)$ and $\psi(x)$ are also studied. Our main results in this connection are the following.

If $f(x)$ is an even and continuous function with period $2\pi$, then the function $\varphi(x)$ is also continuous in $[-\pi, \pi]$.

If $f(x)$ is an even and continuous function with period $2\pi$ and

\[
f(0) = \int_0^\pi f(x) \, dx = 0,
\]

then the function $\psi(x)$ is continuous on $[-\pi, \pi]$; if the condition (2) is not satisfied, then, in general, the function $\psi(x)$ is discontinuous at the point 0.

If $f(x)$ is an even and bounded function with period $2\pi$ and the partial sums of series $\sigma[f]$ are bounded at the point 0, then the partial sums of the series $\sigma[\varphi]$ will be uniformly bounded on $[-\pi, \pi]$. If, on the other hand, instead of the boundedness of the function $f(x)$ we require continuity, then the series $\sigma[\varphi]$ will converge uniformly on $[-\pi, \pi]$ to $\varphi(x)$ (by the $(C, \alpha)$ method for arbitrary $\alpha > -1$). A similar problem is studied for the series $\sigma[\psi]$ as well.

It is shown that these assertions become invalid in the case when the original function $f(x)$ is odd. This indicates that the corresponding assertions of M. and S. Izumi [6] are in error.

The third chapter examines problems relating to the properties of conjugate functions of two variables:

\[
\tilde{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^\pi f(x + s, y) \, \text{ctg} \frac{s}{2} \, ds,
\]

\[
\tilde{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^\pi f(x, y + t) \, \text{ctg} \frac{t}{2} \, dt,
\]

\[
\tilde{f}_3(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^\pi \int_{-\pi}^\pi f(x + s, y + t) \, \text{ctg} \frac{s}{2} \, \text{ctg} \frac{t}{2} \, ds \, dt.
\]

The main results in this connection are assertions which relate, respectively, to the problem of the existence of the function $\tilde{f}_2(x, y)$ and to the summability of the functions $\tilde{f}_i(x, y)$ ($i = 1, 2, 3$).

It is shown that for an arbitrary sequence $\{e_m\}_{m=1}^\infty, \; e_m > 0, \; \lim_{m \to \infty} e_m = 0$ and an arbitrary $\eta \in [0, 1)$ there exists a function $f(x, y) \in L(\log^L)^n$, which has a period of $2\pi$ and for which, on a set with a complete plane measure,

* Depending on the evenness or oddness of $f(x)$, the functions $\varphi(x)$ and $\psi(x)$ are defined appropriately on the interval $[-\pi, 0]$.