A Lagrangian 'prescription' for Landshoff’s alpha prescription in the temporal gauge

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Abstract. I discuss the Landshoff alpha prescription for the propagator pole ambiguity in temporal gauge Yang-Mills theory. The prescription may be generated from a Lagrangian which includes an extra term that is explicitly gauge invariant. No explanation for this term is yet understood. Some of the positive and negative features of this extra term are highlighted. In particular it is consistent with a decoupling of Faddeev-Popov ghosts.

This paper is concerned with the gauge pole problem in the temporal gauge in Yang-Mills (YM) theory. An extensive introduction to the history of this issue in studies of noncovariant gauges can be found in the review of Leibbrandt [1]. An indication of some of the work in the field can be found in the proceedings of a workshop on the problem [2] and a coherent presentation of at least one point of view can be found in the book by Bassetto et al. [3]. Here I shall address the alpha prescription suggested by Landshoff [4] to solve this problem. In particular I shall show that the alpha prescription may be generated from a BRST Lagrangian approach. I cannot yet give an explanation for the origin of a new term I add to the YM Lagrangian but this term is necessary to generate the alpha prescription from a manifestly BRST invariant formalism.

To state the essence of the initial problem: if one considers YM theory in a noncovariant gauge defined by the homogeneous condition

\[ n \cdot A^a = 0 \]  

at least in the perturbative framework one is confronted with the problem of giving meaning in momentum integrations to the spurious pole at \( p \cdot n = 0 \) in the gauge-field propagator

\[ D_{\mu \nu}^{ab}(p) = \frac{\delta^{ab}}{p^2} \left[ g_{\mu \nu} - \frac{p_{\mu} n_{\nu} + p_{\nu} n_{\mu}}{n \cdot p} + \frac{n^2 p_{\mu} p_{\nu}}{(p \cdot n)^2} \right]. \]  

(2)

I shall be interested in the case where \( n_a \) is timelike (\( n^2 > 0 \)).

The alpha prescription [4] is the least studied of prescriptions in the temporal gauge (for an outline of other prescriptions see [1] and [3]). Here the regulated propagator takes the form

\[ D_{\mu \nu}^{ab}(p) = \frac{\delta^{ab}}{(p^2 + i\epsilon)} \left[ g_{\mu \nu} - \frac{n \cdot p (p \cdot n p_{\mu} + p_{\nu} n_{\mu})}{(p \cdot n)^2 + \alpha^2 (n^2)^2} \right. 
\left. + \frac{n^2 p_{\mu} p_{\nu} - \alpha^2 n^2 n_{\mu} n_{\nu}}{(p \cdot n)^2 + \alpha^2 (n^2)^2} \right]. \]  

(3)

The limit \( \alpha \to 0 \) restores the original (2) but the form (3) has the additional feature of satisfying algebraically and for arbitrary \( \alpha \) the 'gauge condition' for a true temporal gauge propagator \( n^a \cdot D_{\mu \nu}(p) = 0 \). This is especially useful in Wilson loop computations. Landshoff has demonstrated that (3) satisfies the Wilson loop criterion without requiring the introduction of Faddeev-Popov (FP) ghost fields. However the taking of the limit \( \alpha \to 0 \) is delicate as can be seen by computing the Fourier transform of the momentum space propagator: the limit generates \( 1/\alpha \) divergences [6]. Landshoff observes that keeping \( \alpha \) nonzero until the very end of the Wilson loop computation enables dangerous \( 1/\alpha \) terms to cancel, making the limit \( \alpha \to 0 \) safe, and leaving behind precisely those terms that give the correct result for this gauge independent quantity*.

The singular behaviour in, at least, gauge dependent Green’s functions suggests the alpha prescription cannot be understood in the sense of distributions. So the parameter \( \alpha \) must appear in the Lagrangian from the outset.

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* Because \( \alpha/\sqrt{n^2} \) carries the dimensions of mass there are also tadpole graphs which contribute. Using a similar approach to Landshoff, these have been shown in [5] to have no effect on the Wilson loop result in the limit \( \alpha \to 0 \).
Two previous approaches are in this spirit [7, 9] but have their respective problems: the former not satisfying the gauge condition except when α = 0, the latter generating an α dependent prescription for the $p^2 = 0$ Feynman pole. In general, the possibility of generating the alpha prescription (3) by any legitimate gauge fixing condition imposed on YM theory is a priori ruled out by the Cheng-Tsai theorem [8], which cannot accommodate the tensor structure $n_u n_v$ in the propagator. Thus the origin of this prescription must lie outside the conventional structure of Yang-Mills + gauge-fixing terms in the Lagrangian.

Consider now the following Lagrangian density which could be said to describe an 'extended' gauge theory:

$$L_{YM} + L_a$$

where $L_{YM}$ is the usual SU(N) Yang-Mills Lagrangian density, $-\frac{1}{4} F^a_{\mu
u} F^{a\mu
u}$, and

$$F^a_{\mu
u} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

is the field strength tensor where $f^{abc}$ are antisymmetric structure constants. The additional Lagrangian density is

$$L_a = -\frac{1}{2} \partial^\mu A^b \frac{1}{\sqrt{N^2 - 1}} F^{a\mu
u} \left( \frac{\alpha^2 n^2}{\text{Tr}(\partial^2)/(N^2 - 1) - \alpha^2 n^2} \right) \times \partial_c \partial^c \frac{1}{\sqrt{N^2 - 1}} F^{a\mu
u}$$

(6)

Here $\partial^\mu A^b$ denotes the covariant derivative in the adjoint representation

$$\partial^\mu A^b = \delta^b_\mu \partial_\mu - g f^{abc} A^c_\mu$$

(8)

By $K^a_\mu$ I mean the 'inverse' of the covariant derivative contracted with the gauge-vector, namely $(n \cdot \partial^\mu)^{-1}$, which I shall expand perturbatively in the coupling constant, $g$:

$$K^a_\mu = \delta^a_\mu (1/n \cdot \partial) + g f^{abc} (1/n \cdot \partial) n^a A^b (1/n \cdot \partial) + \ldots$$

(8)

In this presentation I shall be cavalier in my treatment of the $1/n \cdot \partial$ factors. In fact the conclusions reached in this treatment do not change if certain subtleties are taken into account, but the interested reader is referred to [6] for these details. The other unusual quantity appearing in the alpha Lagrangian is the colour trace (Tr) of the square of the covariant derivative suitably normalised by $\delta^{ab} \delta^{cd} = N^2 - 1$ in the adjoint representation:

$$\text{Tr}(\partial^2)/(N^2 - 1) - \alpha^2 n^2$$

(9)

where the Casimir is defined by $f^{abc} f^{a\mu
u} f^{b\mu
nu} = C_2 (G)/(N^2 - 1)$$

I observe a number of features in (6). The most important aspect is that (6) is itself gauge invariant and thus trivially BRST invariant, hence the term 'extended gauge theory' used earlier to describe the total Lagrangian (4). For the bare Lagrangian $L_a$ the limit $\alpha \to 0$ is safe and it vanishes in that limit restoring the original YM theory. $L_a$ also breaks Lorentz invariance (via the presence of the gauge-vector $n_u n_v$). But, again, in the limit $\alpha \to 0$ the Lorentz symmetry is restored. One may hope that the quantised theory will respect these properties.

The gauge invariance of the extended theory means we can apply the standard FP procedure to quantise. Choosing the homogeneous axial gauge, we find that FP ghosts decouple in the usual manner [1]. Thus the FP method gives an effective Lagrangian

$$L_{eff} = L_{YM} + L_a - (1/2\lambda)(n \cdot A)^2$$

(10)

I shall only be interested in the limit $\lambda \to 0$, which corresponds with the homogeneous condition (1).

The kinetic term from (10) gives the following momentum space gauge-field propagator:

$$D_{\mu\nu}(p) = \frac{\delta^{ab}}{p^2} \left[ g_{\mu\nu} - \frac{g_{\mu\nu} p_{\alpha} p_{\beta} + p_{\mu} p_{\nu}}{(p^2)^2 + \lambda^2 (n^2)^2} \right]$$

(11)

We observe that the $p^2$ denominator is generated without any alpha dependent prescription – unlike the approach in [9]. We are free to invoke the Feynman prescription for this pole. We need not register alarm at the appearance of an unprescribed $1/(p^2)$ term since the homogeneous case ($\lambda = 0$) is all we shall be concerned with (though a different form for the temporal gauge fixing Lagrangian can give a prescription for this term as well). Taking the limit $\lambda \to 0$ in (11) gives exactly (3).

Turning to the other Feynman rules in the theory, the YM Lagrangian yields the usual 3-gluon and 4-gluon vertices. The non-Abelian part of the alpha Lagrangian demand new vertices in the theory, a direct consequence of the retention of BRST invariance with (6). The new vertices fall into two groups. The first group is seen to decouple: Some of these vertices vanish identically by the antisymmetric structure constants being contracted with a colour symmetric piece. The remainder in this group are proportional to the gauge vector $n_u$ for which diagrams contributing to physical processes (namely with gluons on all legs of the vertex) will vanish by $n^a D_{\mu\nu}(p) = 0$.

On the other hand there is no avoiding the second group of vertices, an infinite number arising from the expansion in the coupling constant, $g$, of

$$\alpha^2 n^2$$

$$p^2 - g^2 (C_2 (G)/(N^2 - 1)) A^2$$

which appears in (6). This gives rise to gluon vertices involving an even number of gluons. By the dimensionality of $\alpha \sqrt{n^2}$ we would not expect these vertices to contribute new ultraviolet divergences; the renormalisation should be no worse as a consequence of these terms. However the vertices can possibly affect the Wilson loop. Consider the lowest order contribution – a four gluon vertex. This is found to be:

$$p_{\mu\nu\rho\delta}(p, q, k, r) = -i \alpha^2 n^2 g^2 C_2 (G)/(N^2 - 1) \delta^{ab} \delta^{cd} [k_\mu]_{p, \rho} [r_\nu]_{q, \sigma} + \text{cyclic perms.}$$

(12)

where

$$[A, B] = AB + BA$$

(13)

$$p_\mu = p_\mu / (p^2 + x^2 n^2)^2$$

(15)

The lowest order contribution to the Wilson loop involving this vertex is (for nonzero $x$) a nonvanishing tadpole graph. Scaling arguments reveal the truncated tadpole graph to be proportional to $x^2$. It remains to be seen whether, when inserted into the static Wilson loop, the