The Gribov-Lipatov Relation in QCD

D.J. Pritchard
Department of Applied Mathematics and Theoretical Physics Cambridge University, Silver Street, Cambridge CB3 9EW, U.K.

Received 4 October 1978, in revised form 20 November 1978

Abstract. We analyse and calculate the structure functions of both deep inelastic leptoproduction and $e^+e^-$ annihilation in QCD and in the parton model in a way which brings out the similarities and differences between the two situations. In particular we give a probability interpretation of the Gribov-Lipatov relation for quarks in QCD. The relation does not hold for hadronic structure functions but we determine its modified consequences for this case. A generalised Drell-Yan conjecture is also discussed and verified to lowest order in QCD.

1. Introduction

The deep inelastic structure functions $W_1, W_2$ are well known objects theoretically, and much time has been devoted to their investigation. The techniques used to evaluate them are well understood, and vary in sophistication from the covariant and asymptotically free parton models (CPM and AFPM) [1] to the full power of the renormalisation group (RG) and operator product expansion (OPE) [2].

Slightly less intensely investigated are a similar set of structure functions $\hat{W}_1, \hat{W}_2$ occurring in $e^+e^-$ annihilation. Here there is no operator product expansion and parton model techniques must be used.

Although the two cases are apparently linked by crossing, and one is tempted to consider the annihilation structure functions as continuations of the leptoproduction functions, it is well known that this is not the case [3]. Our techniques apply equally to both situations and a relationship between them is found in the leading logarithm approximation, namely

$$W(\omega) = -\omega^{-1}\hat{W}(\omega^{-1}) \quad (1.1)$$

This result, originally due to Gribov and Lipatov [4] in a slightly different model, is derived in QCD and refers to quark scattering processes. When hadrons are considered (1.1) has to be modified. Instead of (1.1) we find a simple relation between the moments of the hadronic structure functions

$$M_n = \int_0^1 d\omega \omega^{-n} W(\omega) \sim (ln q^2)^{-\alpha_n} M_n^{(0)}$$

$$\bar{M}_n = \int_0^1 d\omega \omega^n \hat{W}(\omega) \sim (ln q^2)^{-\alpha_n} \bar{M}_n^{(0)} \quad (1.2)$$

The exponents $\alpha_n$ are the same function of $n$ in the two equations, but the coefficients $M_n^{(0)}$ and $\bar{M}_n^{(0)}$ are not simply related.

We show that these results have a simple interpretation in terms of the probability distributions of Altarelli and Parisi [5], extended to the annihilation region. The Gribov-Lipatov (GL)-relation is found to show that the probability of finding a quark of given momentum fraction “inside” another quark is independent of the process being considered. We conclude that the GL relation must hold in any theory with such a physical interpretation as that of Altarelli and Parisi.

In Sect. 2, which is a kinematic preliminary to the later sections, we define the structure functions and proceed as far as the GL relation in QCD. In section 3 we give a new interpretation of the results of section 2 in the style of Alterelli & Parisi. Section 4 obtains the consequences for hadrons of the results quarks given in the previous sections.

In Sect. 5 we apply the techniques of Sect. 2 and 4 to generalise, and verify to first order in the QCD coupling $\alpha_s$, the Drell-Yan conjecture to a variety of two hadron-one photon ($=$ two lepton) processes.

2. The GL Relation in QCD

We define the structure functions as a sum, over states $X$ and hadron polarisations, of the square of the matrix elements illustrated in Fig. 1. All the particles in $X$ are to be in the final state, but the photon and hadron may be in the initial or final states. The hadron and the particles of $X$ are to be
on mass-shell, but the photon need not be. Usually in discussing the leptoproduction structure functions one uses the optical theorem to relate this to the imaginary part of a physical forward elastic amplitude. That approach is inapplicable in the annihilation region, and will not be used here.

We therefore define

$$W^u = \Sigma \bar{u}(p) A^u(p, q) u(p)$$

$$= \text{Tr} \left[ (p + m) A^u(p, q) \right]$$ (2.1)

where $A^u$ is the amplitude of Fig. 1, without the initial and final spinors or photon polarization vectors. Spinors are normalised so that $\bar{u} u = 2m$.

We can use current conservation, $qA^u = 0$, to simplify the tensor structure of $W^u$ to one involving only two unknown scalar functions. These are the required structure functions, and can be defined several ways.

$$W^u = -\left( g_{u^v} - \frac{g_{u^v} q^2}{q^2} \right) W_1$$

$$+ \frac{1}{M^2} \left( p' - \frac{p' q^2}{q^2} q^v \right) \left( p - \frac{p q^2}{q^2} q^v \right) W_2$$ (2.2a)

$$= -g_{1^v} W_T + \frac{\alpha}{\sqrt{2}} \left( p' q - \frac{v}{q^2} q^v \right) \left( p - \frac{v}{q^2} q^v \right) W_L$$ (2.2b)

where

$$-g_{1^v} = -g_{u^v} + \frac{p' q - q^v p^v}{p' q} - \frac{q^2 p^v}{(p' q)^2}$$

is the sum over the two independent transverse photon polarizations, and

$$W_T = W_1$$

$$W_2 = W_1 + \frac{(p' q)^2}{q^2 M^2} W_2$$ (2.3)

The structure is reduced still further in the leading logarithm approximation, because the Callan-Gross relation holds, i.e. $W_L \sim 0$. We can therefore write

$$W^v(p, q) = (-g_{1^v}) W(p, q)$$ (2.4)

These structure functions are defined as part of some physical total or semi-inclusive cross-section.

Consequently we are only interested in the region

$$M^2 = (p + q)^2 > M^2, \quad (p + q)^0 > 0$$ (2.5)

since the missing component of the final state must contain at least one hadron, and have positive energy. We can distinguish 3 physically distinct cases subject to (2.5), which are discussed in App. A., but we concern ourselves with only two of these.

1) $v > 0, q^2 < 0, \omega > 1$ for leptoproduction

2) $v < 0, q^2 > 0, 0 < \omega < 1$ for annihilation

We now take as our basic theory that of Quantum-Chromo-Dynamics (QCD) with a non-Abelian SU3 colour gauge field interacting with 6 flavours of quarks. The modifications necessary for any (almost) point coupled theory such as AFPM will be obvious [5, 15]. The extension of our ideas to hadrons will be discussed in Section 4.

Our analysis follows that of Gribov and Lipatov [4], and of Llewellyn-Smith [10], in many respects, but we set it in out in a way which makes apparent the similarities and differences between the two regions.

We define [4] two null vectors $p'$ and $q'$ implicitly by

$$p = p' + \frac{m^2}{2 p \cdot q} q'$$

$$q = q' + \frac{q^2}{2 p \cdot q} p'$$ (2.6)

the use of which simplifies the algebra considerably.

We shall use an axial gauge, in which the gluon propagator is

$$D^{\mu\nu}(k) = \frac{-k_{\mu} q_{\nu} + q_{\mu} k_{\nu}}{k^2 + i \epsilon}$$

(2.7)

which manifestly satisfies the conditions

$$k_{\mu} D^{\mu\nu}(k) = 0 (k^2), \quad q_{\mu} D^{\mu\nu}(k) = 0$$ (2.8)

Again this simplifies the problem considerably, for, as discussed in greater detail in [6] and [10] we only need to consider planar graphs. As we are more concerned with the relationship between leptoproduction than a detailed calculation of either, we need only include such detail as are relevant to the physics.

If we consider Fig. 2, with $p_1^2 \ll p_2^2$, we have the