HOMOLOGICAL ALGEBRA IN PRE-ABELIAN CATEGORIES

A. V. Yakovlev

We construct derived functors in additive categories in which each morphism has a kernel, cokernel, image, and coimage, but the image and coimage are not necessarily isomorphic. We prove that these derived functors possess the usual properties. The main difficulty is that the $3 \times 3$-lemma does not necessarily hold in the categories under consideration.

Recently the essential role played by pre-Abelian categories in the study of various questions in representation theory has been clarified. In particular, these categories arise naturally in the representation theory of nonsemisimple algebras, quivers, partially ordered sets, etc. Of great importance in representation theory is the use of homological algebra, particularly the functors $\text{Ext}$. However, the usual construction of such functors does not carry over to pre-Abelian categories. The aim of the present paper is to construct derived functors in pre-Abelian categories and prove the usual properties.

1. Pre-Abelian Categories

Recall (see [1, Chap. 5, Sec. 1]) that an additive category is called pre-Abelian if for each pair of objects there exists their sum (which is also their product) and if for any morphism there exist a kernel and cokernel (and, therefore, an image and coimage). Since we will repeatedly use the definitions of kernel, cokernel, image, and coimage, we reproduce them here. If $\alpha: A \to B$ is a morphism of a pre-Abelian category, its kernel $\text{Ker}\alpha$ is a pair $(\mathcal{K}, i)$ consisting of an object $\mathcal{K}$ and a morphism $i: \mathcal{K} \to A$ such that $ai = 0$ and, for any morphism $f: X \to A$ such that $af = 0$, there exists a unique morphism $f': X \to \mathcal{K}$ such that $fi = f'$. The cokernel $\text{Coker}\alpha$ is defined dually and is a pair $(\mathcal{M}, j)$ consisting of an object $\mathcal{M}$ and a morphism $j: B \to \mathcal{M}$. Suppose $(\mathcal{K}, i) = \text{Ker}\alpha$, $(\mathcal{M}, j) = \text{Coker}\alpha$; then, by definition, $\text{Coim}\alpha = \text{Coker}i$, $\text{Im}\alpha = \text{Ker}j$.

Recall also (see [1, Proposition 5.6]) that for each morphism $\alpha: A \to B$ of a pre-Abelian category there exists a canonical decomposition

$$A \to \text{Coim}\alpha \overset{\alpha}{\longrightarrow} \text{Im}\alpha \to B.$$ 

A morphism $\alpha$ will be called strict if $\alpha$ is an isomorphism. A pair consisting of an object $C$ and a strict monomorphism $\alpha: C \to A$ will be called a strict subobject of $A$, and a pair consisting of an object $\mathcal{Q}$ and a strict epimorphism $\beta: A \to \mathcal{Q}$ will be called a strict quotient object of $A$.

Some of the following lemmas and propositions contain two dual assertions; we will prove only the first assertion in such cases.

**Lemma 1.** Suppose $\alpha: A \to B$ is a morphism of a pre-Abelian category, $(\mathcal{K}, i)$ its image, and $(\mathcal{M}, j)$ its coimage. Then $\text{Ker}j = \text{Ker}\alpha$, $\text{Coker}i = \text{Coker}\alpha$.

**Proof.** Let $\lambda: M \to B$ denote the canonical morphism $\text{Coim}\alpha \to \text{Im}\alpha \to B$ (so that $\alpha = \lambda j$). Suppose $\text{Ker}\alpha = (K, l)$; by definition of coimage, $\text{Coim}\alpha = \text{Coker}l$, hence $jl = 0$. If $\gamma: X \to A$ is a morphism such that...
Proposition 1. If \( \alpha : A \to B \) is any morphism, then its kernel is a strict subobject of \( A \) and its cokernel a strict quotient object of \( B \). Any strict subobject is the kernel of some morphism (and even some strict epimorphism), and any strict quotient object is the cokernel of some morphism (and even some strict monomorphism).

Proof. In the notation of Lemma 1, the morphism \( l \) defining the kernel \( (K, l) \) is a monomorphism (see [1, Proposition 2.13]) and therefore \( \text{Coim} l = (K, 1_K) \), where \( 1_K \) is the identity automorphism of \( K \). Since \( Jm l = Ker j \) coincides, according to Lemma 1, with \( \text{Ker} \alpha = (K, l) \), the canonical decomposition of \( l \) has the form

\[
K \overset{l}{\longrightarrow} K \overset{\overline{l}}{\longrightarrow} K \overset{\alpha}{\longrightarrow} A.
\]

Clearly, \( \overline{l} \) is the identity isomorphism, hence \( l \) is a strict monomorphism.

Conversely, suppose \( (K, l) \) is a strict subobject of \( A \) and \( (C, j) = \text{Coker} l \). Since \( l \) is a monomorphism, its coimage is \( (K, 1_K) \), and it follows from the strictness of \( l \) that \( Jm l = (k, l) \). But by definition, \( Jm l = Ker j \), i.e., the strict subobject \( (K, l) \) is the kernel of the strict epimorphism \( j \). Proposition 1 is proved.

If \( (K, l) \) is a strict subobject of \( A \) and \( (C, j) = \text{Coker} l \), then \( (C, j) \) is a strict quotient object of \( A \) (Proposition 1). We will denote \( C \) by \( A/K \) and call \( j \) a canonical morphism of the object \( A \) onto the quotient object \( A/K \). It is clear from the proof of Proposition 1 that \( (K, l) = \text{Ker} j \), \( (A/K, j) = \text{Coker} l \).

A sequence

\[
A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C
\]

of objects and morphisms of a pre-Abelian category will be called exact at \( B \) if \( \alpha \) and \( \beta \) are strict morphisms and \( Jm \alpha = Ker \beta \).

Proposition 2. A sequence

\[
C \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C
\]

is exact if and only if \( \beta \) is a strict morphism and \( (A, \alpha) = \text{Ker} \beta \). A sequence

\[
A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\]

is exact if and only if \( \alpha \) is a strict morphism and \( (C, \beta) = \text{Coker} \alpha \).

Proof. Suppose \( (A, \alpha) = \text{Ker} \beta \). Then \( \alpha \) is a monomorphism and \( \text{Ker} \alpha = (0, \theta) \), where \( \theta \) is the unique morphism \( 0 \to A \). Obviously, \( \text{Coker} \theta = (A, 1_A) \), hence \( Jm \theta = Ker 1_A = (0, \theta) = Ker \alpha \), i.e., sequence (1) is exact at \( A \). Also, \( \text{Coim} \alpha = \text{Coker} \theta = (A, 1_A) \), and it follows from the strictness of \( \alpha \) that \( Jm \alpha = (A, \alpha) = Ker \beta \), which means that sequence (1) is exact at \( B \).

Now suppose sequence (1) is exact. It follows from exactness at \( A \) that \( Ker \alpha = Jm \theta = (0, \theta) \) (see the beginning of the proof of this proposition). Thus, \( \alpha \) is a monomorphism. Also, \( \text{Coim} \alpha = \text{Coker} \theta = (A, 1_A) \), and it follows from the strictness of \( \alpha \) that \( Jm \alpha = (A, \alpha) \). But sequence (1) is exact at \( B \), hence \( Ker \beta = Jm \alpha = (A, \alpha) \).

COROLLARY. A sequence

\[
0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\]

is exact if and only if \( (A, \alpha) = \text{Ker} \beta \), \( (C, \beta) = \text{Coker} \alpha \), or, equivalently, if and only if \( (A, \alpha) \) is a strict subobject of \( B \), \( C = B/A \), and \( \beta \) is a canonical epimorphism of \( B \) onto \( B/A \).