The Consistency of the Chiral Schwinger Model

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Abstract. The Chiral Schwinger Model is solved using path integral methods. It is shown that the theory has a consistent solution despite the presence of a chiral anomaly if a suitable regularization procedure is used.

Gauge theories with a chiral coupling of the fermions to the vector boson are well known to be anomalous [1]. That is, the gauge current which is conserved in the classical theory acquires a non-zero divergence in the quantum theory. This anomaly endangers the consistency of the theory because the field equations

\[ \partial_\mu F^{\mu\nu} = j^\nu \]

require

\[ \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0 \]

while the anomaly produces

\[ \partial_\nu j^\nu \neq 0. \]

This quantum mechanical breaking of (chiral) gauge symmetries has led to the absence of anomalies being imposed as a strict condition on realistic physical theories [2]. Recently, however, it has been claimed [3] that a chiral U(1) gauge theory in two dimensions (the Chiral Schwinger Model) may yield a consistent theory despite the presence of the anomaly. An exact solution of this model using path integral methods is given below, showing that a consistent theory can be obtained.

The generating functional of the Chiral Schwinger Model is

\[ Z = N \int DA_\mu D\psi D\bar{\psi} \exp(i \frac{1}{2} d^2 x \mathcal{L}) \]

with

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \left[ \partial_\mu - ie \frac{1}{2} (1 - \gamma_5) A_\mu \right] \psi. \]

It has been long established that the model defined by the classical lagrangian

\[ \mathcal{L}' = \mathcal{L} + \frac{1}{2} m_0^2 A_\mu A^\mu \]

yields a consistent quantum theory, provided \( m_0 > e^2 / 4\pi \) [4]. The issue addressed here is whether such a term has to be introduced by hand or whether it can be obtained by use of a suitable regularization prescription. The claim of [3] is that a consistent solution of the massless theory exists. This solution contains two degrees of freedom, one of which is massless while the other has a mass

\[ m^2 = \frac{e^2}{4\pi} \left( \frac{a^2}{a - 1} \right) \]

where \( a \) is an arbitrary parameter which depends on the regularization scheme used. For \( a > 1 \) the theory is consistent and unitary. The solution depends on abandoning the requirement of gauge invariance which is known to be broken in the quantum theory anyway. The appearance of a regularization dependent mass is, therefore, not surprising since it is common in two dimensional models where the gauge principle is absent [5]. The loss of gauge invariance also accounts for the appearance of the “extra” degree of freedom [3]. A previous investigation of the Chiral Schwinger Model by path integral methods [6] used a gauge invariant regularization procedure and so failed to find a consistent solution.

The solution uses two important relations which occur in two dimensions. Firstly, the vector current is dual to the axial vector

\[ \gamma^\mu \gamma_5 = \gamma^\nu \epsilon^{\nu\mu}. \]
Hence the lagrangian (1) can be written as
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \left[ \partial_\mu - \frac{i}{2} e A_\mu^- \right] \psi \]
where
\[ A_\mu^\pm = (g_\mu^\pm e_\mu) A^\tau. \tag{3} \]
Secondly, the vector field can be written as
\[ A_\mu = -(1/e) (\partial_\mu \sigma + \epsilon_\mu \vec{\sigma} \rho). \tag{4} \]
The generating functional is then
\[ Z = N \int D\sigma D\rho D\psi D\bar{\psi} \exp(i \int d^2 x \mathcal{L}) \]
with
\[ \mathcal{L} = -\frac{1}{2} e^2 \rho \Box \rho + i \bar{\psi} \gamma^\mu \partial_\mu \chi - \bar{\psi} \chi. \tag{5} \]

Following the usual path integral procedure for the solution of two dimensional field theories [7], new fermion variables are defined:
\[ \psi = \exp\left[ -\frac{i}{2} (1 - \gamma_5) (\sigma - \rho) \right] \chi \]
\[ \bar{\psi} = \bar{\chi} \exp\left[ \frac{i}{2} (1 + \gamma_5) (\sigma - \rho) \right]. \tag{6} \]
In terms of the new variables the lagrangian reads
\[ \mathcal{L} = -\frac{1}{2} e^2 \rho \Box \rho + i \bar{\chi} \gamma^\mu \partial_\mu \chi \tag{7} \]
However, the fermionic measure is not invariant under such a change of variables [8]
\[ D\psi D\bar{\psi} = D\chi D\bar{\chi} \cdot J_F. \tag{8} \]

The usual procedure for calculating the jacobian, \( J_F \), is to expand the fermion fields in terms of a set of basis vectors which are eigenstates of the gauge invariant Dirac operator appearing in the lagrangian. However, as the gauge invariance is known to be broken in the quantum theory, the basis states may be chosen to be eigenstates of the gauge variant Dirac operator
\[ D_\mu = \left[ \partial_\mu - ie(\xi g_\mu + \eta e_\mu) A^\tau \right] \tag{9} \]
where \( \xi + \eta = 1 \) to maintain Lorentz covariance [5].
Writing
\[ \xi = \frac{1}{2} (1 + a), \]
\[ \eta = \frac{1}{2} (1 - a) \]
with \( a \) arbitrary, the Dirac operator (9) is
\[ D_\mu = (\partial_\mu - \frac{1}{2} i e \vec{A}_\mu) \tag{10} \]
where
\[ \vec{A}_\mu = \left[ (1 + a)g_\mu - (1 - a)e_\mu \right] A^\tau = A_\mu^- + a A_\mu^+. \tag{11} \]
The gauge variant regularization thus introduces an arbitrary amount of coupling to fermions of the opposite chirality. The jacobian is then calculated by the standard method [7] with the result
\[ \ln J_F = (ie/8\pi) \int d^2 x(\sigma - \rho) e^{8\pi \sigma} \bar{\varphi}_I \]
\[ + (1 - a) \sigma \Box (\sigma - 2\sigma \Box \rho). \tag{12} \]
Since the fermions have decoupled, the integration over the fermionic variables \( \chi \) and \( \bar{\chi} \) can easily be performed yielding a trivial determinant which can be absorbed in the normalization. The effective “bosonized” lagrangian for the Chiral Schwinger Model is then
\[ \mathcal{L} = -\frac{1}{2} e^2 \rho \Box \rho + (1/8\pi) \left\{ [(1 + a) \Box \rho - e^2 \Box \sigma = 0, \right. \]
\[ \left. \Box \rho = (1 - a) \Box \sigma \right\} \tag{14} \]
This lagrangian gives the following coupled field equations
\[ \Box \rho + \frac{e^2}{4\pi} (1 + a) \Box \sigma - \frac{e^2}{4\pi} \Box \rho = 0, \tag{15} \]
\[ \Box \sigma = (1 - a) \Box \sigma \tag{16} \]
which have the solution
\[ \left[ \Box + \frac{e^2}{4\pi} \left( \frac{a^2}{a - 1} \right) \right] \Box \rho = 0. \tag{17} \]
For \( a > 1 \), this is the equation of motion for a scalar particle with a mass given by
\[ m^2 = \frac{e^2}{4\pi} \left( \frac{a^2}{a - 1} \right). \]
Equation (16) implies
\[ \rho = (1 - a) \sigma - h \tag{18} \]
where \( h \) is a harmonic function satisfying \( \Box h = 0 \). Substituting (18) into the expression (4) for the vector field gives
\[ A_\mu = -(1/e) (\partial_\mu \sigma + (1 - a) \epsilon_\mu \vec{\sigma} - \epsilon_\mu \vec{h}) \]
which is the result of [3].